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Primordial vorticity and gradient expansion

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Abstract

The evolution equations of the vorticities of the electrons, ions and photons in a pre-decoupling plasma are derived, in a fully inhomogeneous geometry, by combining the general relativistic gradient expansion and the drift approximation within the Adler-Misner-Deser decomposition. The vorticity transfer between the different species is discussed in this novel framework and a set of general conservation laws, connecting the vorticities of the three-component plasma with the magnetic field intensity, is derived. After demonstrating that a source of large-scale vorticity resides in the spatial gradients of the geometry and of the electromagnetic sources, the total vorticity is estimated to lowest order in the spatial gradients and by enforcing the validity of the momentum constraint. By acknowledging the current bounds on the tensor to scalar ratio in the (minimal) tensor extension of the ΛCDM paradigm the maximal comoving magnetic field induced by the total vorticity turns out to be, at most, of the order of 10^{-37} G over the typical comoving scales ranging between 1 and 10 Mpc. While the obtained results seem to be irrelevant for seeding a reasonable galactic dynamo action, they demonstrate how the proposed fully inhomogeneous treatment can be used for the systematic scrutiny of pre-decoupling plasmas beyond the conventional perturbative expansions.

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1 Primordial vorticities?

The observed Universe might originate from a strongly coupled electromagnetic plasma existing prior to photon decoupling where the angular momentum transfer between ions, electrons and photons in an expanding space-time geometry leads to the formation of large-scale vortices as speculated by various authors including, with slightly different perspectives, Hoyle [1], Harrison [2, 3, 4], Mishustin and Ruzmaikin [5], Ozernoy and Chernin [6, 7, 8] and others (see also [9]). The primordial vorticity, if present in the pre-decoupling plasma, might lead eventually to the formation of large-scale magnetic fields possibly relevant for galactic magnetogenesis.

The physical description of the angular momentum exchange between ions, electrons and photons can be realized by appropriately translating the evolution equations describing ionized gases [10, 11] to an expanding geometry supplemented by its own relativistic fluctuations [12, 13]. In the context of the ΛCDM paradigm[3], it is both reasonable and justified to assume that the background geometry is conformally flat and that its inhomogeneities stem from the relativistic fluctuations of the spatial curvature described either in gauge-invariant terms or in an appropriate gauge. The latter assumption rests exactly on the absence of large-scale vorticity which is assumed to be vanishing at least within the current observational precision.

A gross argument could suggest that the vorticity must be negligible for ΛCDM initial conditions, since it is the curl of a velocity. Thanks to the momentum constraint (connecting the first derivatives of the linearized fluctuations of the geometry to the peculiar velocities), the total velocity field is subleading when compared with the density contrasts or with the curvature perturbation for typical scales larger than the Hubble radius and in the case of the conventional adiabatic initial conditions postulated in the vanilla ΛCDM scenario. The latter argument suggests that the treatment of large-scale vorticity assumes, more or less tacitly, a correct treatment of the spatial gradients. To transform this incomplete observation in a more rigorous approach it is necessary to introduce a description of the vorticity which does not rely on the purported smallness or largeness of the gravitational fluctuations. It is rather desirable to describe the angular momentum exchange between ions, electrons and photons in a gravitating plasma which is also fully inhomogeneous. By fully inhomogeneous plasma we mean the situation where not only the concentrations of charged and neutral species depend, in an arbitrary manner, upon the spatial coordinates but where the geometry as well as the electromagnetic fields are not homogeneous. It has been recently argued [14] that such a description can be rather effective for the analysis of a wide range of phenomena including the physics of pre-decoupling plasmas. In the present paper the results of Ref. [14] shall be first extended and then applied to a concrete situation with the purpose of obtaining an explicit set of equations describing the evolution of the vorticities of the various species of the plasma. The proposal of [14] is built on the fully inhomogeneous description of the geometry

[3]The acronym ΛCDM (where Λ denotes the dark energy component and CDM stands for cold dark matter) and the terminology concordance paradigm will be used interchangeably.

in terms of the Adler-Misner-Deser (ADM) variables [15, 16] which are customarily exploited for the implementation of the general relativistic gradient expansion [17, 18, 19, 20, 21, 22]. The second key ingredient of Ref. [14] is the fully inhomogeneous description of cold plasmas in flat space which is the starting points of the analysis of nonlinear effects in kinetic theory and in magnetohydrodynamics (see, e.g. [23, 24, 25]). Consequently, the vorticity exchange between ions, electrons and photons can be analyzed in gravitating plasmas with the help of an expansion scheme which involves not only the gradients of the geometry, but also the gradients of the electromagnetic sources, by so combining the general relativistic gradient expansion and the drift approximation (sometimes dubbed giuding center approximation) typical of cold plasmas.

The approach pursued in this paper reproduces, in the conformally flat limit, the conventional treatment which will be made more precise in section 2. The evolution of the vorticity in gravitating plasmas which are also fully inhomogeneous will be discussed in section 3. In sections 4 and 5 the total vorticity of the geometry will be computed within the gradient expansion and estimated in the framework of the ΛCDM paradigm. The maximal magnetic field induced by the total vorticity will be computed in section 6. Section 7 contains our concluding remarks. In appendix A some useful complements have been included to make the paper self-contained while in appendix B useful details on the calculations of correlation functions of multiple fields in real space have been included for the technical benefit of the interested readers.

2 Vorticities in conventional perturbative expansions

The treatment proposed here differs slightly from the one of Refs. [2, 3, 4, 5] for three reasons: (i) the conformal time coordinate is preferred to the cosmic time; (ii) the relativistic fluctuations of the geometry are included in the longitudinal gauge; (iii) the three-fluid, two-fluids and one fluid descriptions are discussed more explicitly within the appropriate temperature ranges where they are applicable.

The conformal flatness of the geometry does not imply the invariance of the system under the Weyl rescaling of the metric. Such a potential symmetry is broken by the masses of the electrons and ions which are crucial in the large-scale evolution of the vorticity. The considerations of the present section can also be formulated in the case of a geometry which is not spatially flat; this is not essential since the subsequent generalizations will automatically include also geometries which are not necessarily spatially flat. Consider first the case of a conformally flat background geometry characterized by a metric tensor $\overline{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ and supplemented by the corresponding relativistic fluctuations which we write in the longitudinal gauge

$$\delta_{\mathrm{s}} g_{00}(\vec{x},\tau) = 2\,a^2(\tau)\phi(\vec{x},\tau), \qquad \delta_{\mathrm{s}} g_{ij}(\vec{x},\tau) = 2\,a^2(\tau)\psi(\vec{x},\tau)\delta_{ij}; \tag{2.1}$$

note that δ_{s} describes a metric perturbation which preserves the scalar nature of the fluctu-

ation since, in the ΛCDM paradigm, the dominant source of inhomogeneity comes from the scalar modes of the geometry. By defining the comoving electromagnetic fields $\vec{E}$ and $\vec{B}$ as well as the comoving concentrations of electrons and ions (i.e. n_{e} and n_{i})

$$\begin{aligned}
&\vec{E}(\vec{x},\tau) = a^2(\tau)\vec{\mathcal{E}}(\vec{x},\tau), \qquad \vec{B}(\vec{x},\tau) = a^2(\tau)\vec{\mathcal{B}}(\vec{x},\tau),\\
&n_{\mathrm{i}}(\vec{x},\tau) = a^3(\tau)\tilde{n}_{\mathrm{i}}(\vec{x},\tau), \qquad n_{\mathrm{e}}(\vec{x},\tau) = a^3(\tau)\tilde{n}_{\mathrm{e}}(\vec{x},\tau), \qquad (2.2)
\end{aligned}$$

Maxwell's equations read

$$\vec{\nabla}\cdot\vec{E} = 4\pi e(n_{\mathrm{i}} - n_{\mathrm{e}}), \qquad \vec{\nabla}\cdot\vec{B} = 0, \qquad (2.3)$$

$$\vec{\nabla}\times\vec{E} = -\partial_\tau\vec{B}, \qquad \vec{\nabla}\times\vec{B} = 4\pi e(n_{\mathrm{i}}\,\vec{v}_{\mathrm{i}} - n_{\mathrm{e}}\,\vec{v}_{\mathrm{e}}) + \partial_\tau\vec{E}. \qquad (2.4)$$

In Eqs. (2.2), (2.3) and (2.4) all the fields are appropriately rescaled so that the resulting equations are formally equivalent to the ones of the flat space-time. The peculiar velocities of the ions, electrons and photons obey the following set of equations[4]

$$\partial_\tau\vec{v}_{\mathrm{e}} + \mathcal{H}\,\vec{v}_{\mathrm{e}} = -\frac{e n_{\mathrm{e}}}{\rho_{\mathrm{e}}\,a^4}[\vec{E} + \vec{v}_{\mathrm{e}}\times\vec{B}] - \vec{\nabla}\phi + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\mathrm{e}}}a\Gamma_{\gamma\,\mathrm{e}}(\vec{v}_\gamma - \vec{v}_{\mathrm{e}}) + a\Gamma_{\mathrm{e\,i}}(\vec{v}_{\mathrm{i}} - \vec{v}_{\mathrm{e}}), \qquad (2.5)$$

$$\partial_\tau\vec{v}_{\mathrm{i}} + \mathcal{H}\,\vec{v}_{\mathrm{i}} = \frac{e n_{\mathrm{i}}}{\rho_{\mathrm{i}}\,a^4}[\vec{E} + \vec{v}_{\mathrm{i}}\times\vec{B}] - \vec{\nabla}\phi + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\mathrm{i}}}a\Gamma_{\gamma\,\mathrm{i}}(\vec{v}_\gamma - \vec{v}_{\mathrm{i}}) + a\Gamma_{\mathrm{e\,i}}\frac{\rho_{\mathrm{e}}}{\rho_{\mathrm{i}}}(\vec{v}_{\mathrm{e}} - \vec{v}_{\mathrm{i}}), \qquad (2.6)$$

$$\partial_\tau\vec{v}_\gamma = -\frac{1}{4}\vec{\nabla}\delta_\gamma - \vec{\nabla}\phi + a\Gamma_{\gamma\mathrm{i}}(\vec{v}_{\mathrm{i}} - \vec{v}_\gamma) + a\Gamma_{\gamma\mathrm{e}}(\vec{v}_{\mathrm{e}} - \vec{v}_\gamma). \qquad (2.7)$$

In Eqs. (2.5)–(2.7) the relativistic fluctuations of the geometry are included from the very beginning in terms of the longitudinal gauge variables of Eq. (2.1); the electron-photon, electron-ion and ion-photon rates of momentum exchange appearing in Eqs. (2.5)–(2.7) are given by[5]:

$$\Gamma_{\gamma\mathrm{e}} = \tilde{n}_{\mathrm{e}}\sigma_{\mathrm{e}\gamma}, \qquad \Gamma_{\gamma\mathrm{i}} = \tilde{n}_{\mathrm{i}}\sigma_{\mathrm{i}\gamma}, \qquad \sigma_{\mathrm{e}\gamma} = \frac{8}{3}\pi\left(\frac{e^2}{m_{\mathrm{e}}}\right)^2, \qquad \sigma_{\mathrm{i}\gamma} = \frac{8}{3}\pi\left(\frac{e^2}{m_{\mathrm{i}}}\right)^2, \qquad (2.8)$$

$$\Gamma_{\mathrm{e\,i}} = \tilde{n}_{\mathrm{e}}\sqrt{\frac{T}{m_{\mathrm{e}}}}\,\sigma_{\mathrm{e\,i}} = \Gamma_{\mathrm{i\,e}}, \qquad \sigma_{\mathrm{e\,i}} = \frac{e^4}{T^2}\ln\Lambda_{\mathrm{C}}, \qquad \Lambda_{\mathrm{C}} = \frac{3}{2e^3}\sqrt{\frac{T^3}{\tilde{n}_{\mathrm{e}}\pi}}. \qquad (2.9)$$

Note that, in Eq. (2.8) and (2.9), T and $\tilde{n}$ are, respectively, physical temperatures and physical concentrations. If the rates and the cross sections would be consistently expressed in terms of comoving temperatures $\overline{T} = aT$ and comoving concentrations $n = a^3\,\tilde{n}$ the corresponding rates will inherit a scale factor for each mass. For instance $a\Gamma_{\mathrm{e\,i}}$ becomes $n_{\mathrm{e}}\,\sqrt{\overline{T}/(m_{\mathrm{e}}a)}\,(e^4/\overline{T}^2)\,\ln\Lambda_{\mathrm{C}}$, if comoving temperature and concentrations are used.

Let us then define the vorticities associated with the peculiar velocities of the various species

$$\vec{\omega}_{\mathrm{e}}(\vec{x},\tau) = \vec{\nabla}\times\vec{v}_{\mathrm{e}}, \qquad \vec{\omega}_{\mathrm{i}}(\vec{x},\tau) = \vec{\nabla}\times\vec{v}_{\mathrm{i}}, \qquad \vec{\omega}_\gamma(\vec{x},\tau) = \vec{\nabla}\times\vec{v}_\gamma, \qquad (2.10)$$

[4] As usual $\mathcal{H} = \partial_\tau\ln a$ and its relation with the Hubble rate is simply $\mathcal{H} = aH$.

[5] Note that T denotes the temperature and Λ_{C} is the Coulomb logarithm [10, 11].

and their corresponding three-divergences:

$$\theta_{\rm e}(\vec{x},\tau) = \vec{\nabla}\cdot\vec{v}_{\rm e}, \qquad \theta_{\rm i}(\vec{x},\tau) = \vec{\nabla}\cdot\vec{v}_{\rm i}, \qquad \theta_{\gamma}(\vec{x},\tau) = \vec{\nabla}\cdot\vec{v}_{\gamma}. \tag{2.11}$$

The evolution equations of the vorticities and of the divergences can be obtained by taking, respectively, the curl and the divergence of Eqs. (2.5)–(2.7) and by using Eqs. (2.3) and (2.4). To simplify the obtained expressions it is useful to introduce the total comoving charge density and the comoving current density.

$$\rho_{\rm q} = e(n_{\rm i} - n_{\rm e}), \qquad \vec{J} = e(n_{\rm i}\vec{v}_{\rm i} - n_{\rm e}\vec{v}_{\rm e}). \tag{2.12}$$

Thus, the evolution of the vorticities and of the divergences of the electrons are, respectively,

$$\begin{aligned}
\partial_{\tau}\vec{\omega}_{\rm e} + \mathcal{H}\,\vec{\omega}_{\rm e} &= \frac{e n_{\rm e}}{\rho_{\rm e}\,a^{4}}\biggl[\partial_{\tau}\vec{B} + (\vec{v}_{\rm e}\cdot\vec{\nabla})\vec{B} + \theta_{\rm e}\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{v}_{\rm e}\biggr] \\
&+ \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\rm e}} a\Gamma_{\gamma\,\rm e}(\vec{\omega}_{\gamma} - \vec{\omega}_{\rm e}) + a\Gamma_{\rm e\,i}(\vec{\omega}_{\rm i} - \vec{\omega}_{\rm e}), \qquad (2.13)\\
\partial_{\tau}\theta_{\rm e} + \mathcal{H}\theta_{\rm e} &= -\frac{e n_{\rm e}}{\rho_{\rm e}\,a^{4}}\biggl[4\pi\rho_{\rm q} + \vec{\omega}_{\rm e}\cdot\vec{B} - 4\pi\vec{v}_{\rm e}\cdot\vec{J} - \vec{v}_{\rm e}\cdot\partial_{\tau}\vec{E}\biggr] - \nabla^{2}\phi \\
&+ \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\rm e}} a\Gamma_{\gamma\rm e}(\theta_{\gamma} - \theta_{\rm e}) + a\Gamma_{\rm e\,i}(\theta_{\rm i} - \theta_{\rm e}), \qquad (2.14)
\end{aligned}$$

Conversely the vorticity and the three-divergence of the ions evolve as:

$$\begin{aligned}
\partial_{\tau}\vec{\omega}_{\rm i} + \mathcal{H}\,\vec{\omega}_{\rm i} &= -\frac{e n_{\rm i}}{\rho_{\rm i}\,a^{4}}\biggl[\partial_{\tau}\vec{B} + (\vec{v}_{\rm i}\cdot\vec{\nabla})\vec{B} + \theta_{\rm i}\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{v}_{\rm i}\biggr] \\
&+ \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\rm i}} a\Gamma_{\gamma\,\rm i}(\vec{\omega}_{\gamma} - \vec{\omega}_{\rm i}) + a\Gamma_{\rm e\,i}\frac{\rho_{\rm e}}{\rho_{\rm i}}(\vec{\omega}_{\rm e} - \vec{\omega}_{\rm i}), \qquad (2.15)\\
\partial_{\tau}\theta_{\rm i} + \mathcal{H}\theta_{\rm i} &= \frac{e n_{\rm i}}{\rho_{\rm i}\,a^{4}}\biggl[4\pi\rho_{\rm q} + \vec{\omega}_{\rm i}\cdot\vec{B} - 4\pi\vec{v}_{\rm i}\cdot\vec{J} - \vec{v}_{\rm i}\cdot\partial_{\tau}\vec{E}\biggr] - \nabla^{2}\phi \\
&+ \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\rm i}} a\Gamma_{\gamma\rm i}(\theta_{\gamma} - \theta_{\rm i}) + a\Gamma_{\rm e\,i}\frac{\rho_{\rm e}}{\rho_{\rm i}}(\theta_{\rm e} - \theta_{\rm i}). \qquad (2.16)
\end{aligned}$$

Finally, the evolution equations for the photons are given by:

$$\begin{aligned}
\partial_{\tau}\vec{\omega}_{\gamma} &= a\Gamma_{\gamma\rm i}(\vec{\omega}_{\rm i} - \vec{\omega}_{\gamma}) + a\Gamma_{\gamma\rm e}(\vec{\omega}_{\rm e} - \vec{\omega}_{\gamma}), \qquad (2.17)\\
\partial_{\tau}\theta_{\gamma} &= -\frac{1}{4}\nabla^{2}\delta_{\gamma} - \nabla^{2}\phi + a\Gamma_{\gamma\rm i}(\theta_{\rm i} - \theta_{\gamma}) + a\Gamma_{\gamma\rm e}(\theta_{\rm e} - \theta_{\rm i}). \qquad (2.18)
\end{aligned}$$

The system described by the set of equations deduced so far will be considered as globally neutral. In particular, prior to photon decoupling, the electron and ion (comoving) concentrations have a common value n_0, i.e. $n_{\rm i} = n_{\rm e} = n_0$ where[6]

$$n_0 = \eta_{\rm b0} n_{\gamma}, \qquad \eta_{\rm b0} = 6.177\times 10^{-10}\biggl(\frac{h_0^2\Omega_{\rm b0}}{0.02258}\biggr)\biggl(\frac{2.725\,{\rm K}}{T_{\gamma 0}}\biggr)^{3}, \tag{2.19}$$

[6]If not otherwise stated the pivotal values of the cosmological parameters will be the ones determined from the WMAP 7yr data alone in the light of the ΛCDM paradigm.

and $T_{\gamma 0}$ is the present value of the CMB temperature determining the concentration of the photons; $\Omega_{\rm b0}$ is the present value of the critical fraction of baryons, while h_0 is the Hubble constant in units of 100 Km/(Mpc × sec). The system of Eqs. (2.13)–(2.18) is coupled with the evolution of the density contrasts of the electrons, ions and photons (i.e. $\delta_{\rm e}$, $\delta_{\rm i}$ and δ_{γ})

$$\delta_{\rm e}' = -\theta_{\rm e} + 3\psi' - \frac{e n_{\rm e}}{\rho_{\rm e} a^4} \vec{E}\cdot\vec{v}_{\rm e}, \qquad \delta_{\rm i}' = -\theta_{\rm i} + 3\psi' + \frac{e n_{\rm i}}{\rho_{\rm i} a^4} \vec{E}\cdot\vec{v}_{\rm i}, \tag{2.20}$$

$$\delta_{\gamma}' = 4\psi' - \frac{4}{3}\theta_{\gamma}. \tag{2.21}$$

Finally the metric fluctuations, the density contrasts and the divergences of the peculiar velocities are both determined and constrained by the perturbed Einstein equations (see, e.g. Eqs. (2.43)–(2.46) in the first article of Ref. [12]). Concerning the system of Eqs. (2.13)–(2.18) two comments are in order:

- Eqs. (2.13)–(2.14) (as well as Eqs. (2.15)–(2.16)) couple together the evolution of the vorticities, the evolution of the divergences and the gradients of the magnetic field; while in the linearized approximation the spatial gradients are simply neglected, in the forthcoming sections the evolution of the vorticity will be studied to a given order in the spatial gradients;

- the electron and ion masses break the Weyl rescaling of the whole system of equations; this aspect can be appreciated by noticing that the prefactor appearing in front of the square brackets at the right hand side of Eqs. (2.13)–(2.14) and Eqs. (2.15)–(2.16) is, respectively, $e/(m_{\rm e} a)$ and $e/(m_{\rm i} a)$.

Equations (2.13)–(2.18) have three different scales of vorticity exchange: the photon-ion, the photon-electron and the electron ion rates whose respective magnitude determines the subleading terms and the different dynamical regimes. By taking the ratios of the two rates appearing at the right hand side of Eqs. (2.13) and (2.15) the following two dimensionless ratios can be constructed[7]:

$$\frac{3\rho_{\rm e}\,\Gamma_{\rm e\,i}}{4\,\rho_{\gamma}\Gamma_{\gamma \rm e}} = \frac{135\,\zeta(3)}{16\,\pi^5}\left(\frac{T}{m_{\rm e}}\right)^{-5/2} \eta_{\rm b0}\,\ln\Lambda_{\rm C} \equiv \left(\frac{T}{T_{\rm e\gamma}}\right)^{-5/2}, \tag{2.22}$$

$$\frac{3\rho_{\rm e}\,\Gamma_{\rm e\,i}}{4\,\rho_{\gamma}\Gamma_{\gamma \rm i}} = \left(\frac{m_{\rm p}}{m_{\rm e}}\right)^{2}\left(\frac{T}{T_{\rm e\gamma}}\right)^{-5/2} \equiv \left(\frac{T}{T_{\rm i\gamma}}\right)^{-5/2}, \tag{2.23}$$

where $\zeta(3) = 1.202...$ and the ion mass has been estimated through the proton mass; the effective temperatures $T_{\rm e\gamma}$ and $T_{\rm i\gamma}$ introduced in the second equality of Eqs. (2.22) and (2.23) are defined as:

$$T_{\rm e\gamma} = m_{\rm e}\,\mathcal{N}^{2/5}\,\eta_{\rm b0}^{2/5}, \qquad T_{\rm i\gamma} = m_{\rm e}^{-1/5} m_{\rm p}^{4/5}\mathcal{N}^{2/5}\,\eta_{\rm b0}^{2/5}, \qquad \mathcal{N} = \frac{270\zeta(3)}{32\,\pi^5}\ln\Lambda_{\rm C}. \tag{2.24}$$

[7] Note that $\rho_{\rm i}$ must simplify when taking the ratio of the two rates in Eq. (2.15).

In explicit terms and for the fiducial set of cosmological parameters determined on the basis of the WMAP 7yr data alone in the light of the ΛCDM scenario [26, 27]

$$T_{\mathrm{e}\gamma} = 88.6 \left(\frac{h_0^2 \Omega_{\mathrm{b}0}}{0.02258}\right)^{2/5} \mathrm{eV}, \qquad T_{\mathrm{i}\gamma} = 36.08 \left(\frac{h_0^2 \Omega_{\mathrm{b}0}}{0.02258}\right)^{2/5} \mathrm{keV}. \tag{2.25}$$

On the basis of Eq. (2.25) there are three different dynamical regimes. When $T > T_{\mathrm{i}\gamma}$ the ion-photon and the electron-photon rates dominate against the Coulomb rate: in this regime the photons, electrons and ions are all coupled together and form a unique physical fluid with the same effective velocity. When $T_{\mathrm{e}\gamma} < T < T_{\mathrm{i}\gamma}$ the electron-photon rate dominates against the Coulomb rate which is anyway larger than the ion-photon rate. Finally for $T < T_{\mathrm{e}\gamma}$ the Coulomb rate is always dominant which means that the ion-electron fluid represents a unique entity characterized by a single velocity which is customarily referred to as the baryon velocity. The effective temperatures $T_{\mathrm{e}\gamma}$ and $T_{\mathrm{e\,i}}$ determine the hierarchies between the different rates and should not be confused with the kinetic temperatures of the electrons and of the ions which coincide approximately with the photon temperature $T_{\gamma} \simeq T_{\mathrm{e}} \simeq T_{\mathrm{i}}$. For instance after matter-radiation equality $(T_{\mathrm{e}} - T_{\gamma})/T_{\gamma} \simeq \mathcal{O}(H/\Gamma_{\mathrm{e}\gamma})$ and $(T_{\mathrm{i}} - T_{\mathrm{e}})/T_{\gamma} \simeq \mathcal{O}(H/\Gamma_{\mathrm{ei}})$ where H is the standard Hubble rate at the corresponding epoch.

Depending on the range of temperatures the effective evolution equations for the vorticities will change. In the regime $T > T_{\mathrm{i}\gamma}$ the Coulomb rate can be neglected in comparison with the Thomson rates and the vorticities of photons, electrons and ions approximately coincide. For $T_{\mathrm{e}\gamma} < T < T_{\mathrm{i}\gamma}$ the Ohm law can be easily obtained from Eq. (2.5) and it is given by

$$\vec{E} + \vec{v}_{\mathrm{e}} \times \vec{B} = \frac{\vec{J}}{\sigma} + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{b}}}\frac{m_{\mathrm{i}}}{e} a^2 \Gamma_{\gamma\mathrm{e}} (\vec{v}_{\gamma} - \vec{v}_{\mathrm{e}}), \tag{2.26}$$

where it has been used that the baryon density $\rho_{\mathrm{b}} = (m_{\mathrm{i}} + m_{\mathrm{e}})\tilde{n}_0$ coincides approximately with the ion density in the globally neutral case and that $n_0 = a^3 \tilde{n}_0$; furthermore, in Eq. (2.26), σ denotes the electric conductivity [13]

$$\sigma = \frac{\omega_{\mathrm{p\,e}}^2}{4\pi a \Gamma_{\mathrm{ei}}}, \qquad \omega_{\mathrm{p}\,e} = \sqrt{\frac{4\pi e^2\, n_{\mathrm{e}}}{m_{\mathrm{e}} a}}, \tag{2.27}$$

expressed in terms of the Coulomb rate and in terms of the electron plasma frequency[8] $\omega_{\mathrm{p\,e}}$. By taking the curl of both sides of Eq. (2.26) the following relation can be easily derived:

$$\vec{\nabla} \times \vec{E} + \vec{\nabla} \times (\vec{v}_{\mathrm{e}} \times \vec{B}) = \frac{\vec{\nabla} \times \vec{J}}{\sigma} + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{b}}}\frac{m_{\mathrm{i}}}{e} a^2 \Gamma_{\gamma\mathrm{e}} (\vec{\omega}_{\gamma} - \vec{\omega}_{\mathrm{e}}). \tag{2.28}$$

Recalling now Eq. (2.3) and (2.4), Eq. (2.28) becomes:

$$\frac{\partial \vec{B}}{\partial \tau} = \vec{\nabla} \times (\vec{v}_{\mathrm{e}} \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4\pi\sigma} - \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{b}}}\, a^2\, \frac{m_{\mathrm{i}}}{e} \Gamma_{\mathrm{e}\gamma} (\vec{\omega}_{\gamma} - \vec{\omega}_{\mathrm{e}}). \tag{2.29}$$

[8]The electron plasma frequency of Eq. (2.27) must not be confused with the vorticity

In the same regime the evolution equation for the vorticities of the ions and of the photons are, up to spatial gradients,

$$\partial_\tau \vec{\omega}_{\rm i} + \mathcal{H}\vec{\omega}_{\rm i} = -\frac{e n_{\rm i}}{\rho_{\rm i} a^4}\partial_\tau \vec{B}, \tag{2.30}$$

$$\partial_\tau \vec{\omega}_\gamma = a\Gamma_{\gamma {\rm e}}(\vec{\omega}_{\rm e} - \vec{\omega}_\gamma). \tag{2.31}$$

By eliminating the electron-photon rate between Eqs. (2.30) and (2.31) and by neglecting the spatial gradients in Eq. (2.29), the following pair of approximate conservation laws can be obtained

$$\partial_\tau\left(a\vec{\omega}_{\rm i} + \frac{e}{m_{\rm i}}\vec{B}\right) = 0, \tag{2.32}$$

$$\partial_\tau\left(\frac{e}{m_{\rm i}}\vec{B} - \frac{a}{R_{\rm b}}\vec{\omega}_\gamma\right) = 0, \tag{2.33}$$

where the ratio $R_{\rm b}$ is given by:

$$R_{\rm b} = \frac{3}{4}\frac{\rho_{\rm b}}{\rho_\gamma} = 30.36\left(\frac{10^3}{z}\right) h_0^2 \Omega_{\rm b0}. \tag{2.34}$$

By further combining the relations of Eqs. (2.32) and (2.33) the vorticity of the photons can be directly related to the vorticity of the ions since $\partial_\tau[R_{\rm b}\vec{\omega}_{\rm i} + \vec{\omega}_\gamma] = 0$. By assuming that at a given time $\tau_{\rm r}$ the primordial value of the vorticity in the electron photon system is $\vec{\omega}_{\rm r}$ and that $\vec{B}(\tau_{\rm r}) = 0$ we shall have that

$$a_{\rm r}\vec{\omega}_{\rm i}(\tau_{\rm r}) + \frac{4}{3}\frac{\rho_\gamma(\tau_{\rm r})}{\rho_{\rm b}(\tau_{\rm r})} a_{\rm r}\vec{\omega}_\gamma(\tau_{\rm r}) = \vec{\omega}_{\rm r}. \tag{2.35}$$

Thus the solution of Eqs. (2.32) and (2.33) with the initial condition (2.35) can be written as:

$$\vec{\omega}_{\rm i}(\vec{x},\tau) = -\frac{e}{m_{\rm i}}\frac{\vec{B}(\vec{x},\tau)}{a(\tau)} + \frac{a_{\rm r}}{a(\tau)}\vec{\omega}_{\rm r}, \tag{2.36}$$

$$\vec{\omega}_\gamma(\vec{x},\tau) = \frac{R_{\rm b}(\tau)}{a(\tau)}[\vec{\omega}_{\rm r} - a(\tau)\vec{\omega}_{\rm i}(\vec{x},\tau)]. \tag{2.37}$$

The approximate conservation laws of Eqs. (2.32)–(2.33) can also be phrased in terms of the physical vorticities $\vec{\Omega}_X(\vec{x},\tau) = a(\tau)\vec{\omega}_X(\vec{x},\tau)$ where X denotes a generic subscript[9].

For typical temperatures $T < T_{{\rm e}\gamma}$ the electrons and the ions are more strongly coupled than the electrons and the photons. This means that the effective evolution can be described in terms of the one-fluid magnetohydrodynamical (MHD in what folllows) equations where,

[9]Note that while $\vec{\omega}_X$ is related to $\vec{B}$, the physical vorticity $\vec{\Omega}_X$ is directly proportional to $\vec{\mathcal{B}}$. For instance, in the treatment of [2, 3, 4] the use of the physical vorticity and of the physical magnetic field is preferred.

on top of the total current $\vec{J}$ the center of mass vorticity of the electron-ion system is introduced

$$\vec{\omega}_{\rm b} = \frac{m_{\rm i}\vec{\omega}_{\rm i} + m_{\rm e}\vec{\omega}_{\rm e}}{m_{\rm e} + m_{\rm i}}. \tag{2.38}$$

Equation (2.13) (multiplied by $m_{\rm e}$) and Eq. (2.15) (multiplied by $m_{\rm i}$) can therefore be summed up with the result that

$$\partial_\tau \vec{\omega}_{\rm b} + \mathcal{H}\vec{\omega}_{\rm b} = \frac{\vec{\nabla}\times(\vec{J}\times\vec{B})}{a^4\rho_{\rm b}} + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm b}} a\Gamma_{\gamma\,{\rm e}}(\vec{\omega}_\gamma - \vec{\omega}_{\rm b}). \tag{2.39}$$

The evolution equation for the total current can be obtained from the difference of Eqs. (2.5) and (2.6). Since the interaction rates are typically much larger than the expansion rates the Ohm equation can be simplified and becomes

$$\vec{E} + \vec{v}_{\rm b}\times\vec{B} = \frac{\vec{J}}{\sigma} + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\rm b}}\frac{m_{\rm i}}{e} a^2 \Gamma_{\gamma {\rm e}}(\vec{v}_\gamma - \vec{v}_{\rm b}), \tag{2.40}$$

where $\vec{v}_{\rm b}$ is the baryon velocity related to the baryon vorticity as $\vec{\omega}_{\rm b} = \vec{\nabla}\times\vec{v}_{\rm b}$. The similarity of Eqs. (2.28) and (2.39) should not be misunderstood: while Eq. (2.28) follows from the right hand side of Eq. (2.5), Eq. (2.39) follows by taking the difference of Eq. (2.6) (multiplied by $n_{\rm i}$) and of Eq. (2.5) (multiplied by $n_{\rm e}$). The expression obtained by means of the latter difference is rather lengthy and can be found in its full generality, in Ref. [13] (see, in particular, Eqs. (7) and (10)). Here the expression has been simplified by neglecting higher orders in $(m_{\rm e}/m_{\rm i})$. The effective set of evolution equations can then be written, in this regime, as

$$\partial_\tau \vec{\omega}_{\rm b} + \mathcal{H}\vec{\omega}_{\rm b} = \frac{\vec{\nabla}\times(\vec{J}\times\vec{B})}{a^4\,\rho_{\rm b}} + \frac{\epsilon'}{R_{\rm b}}(\vec{\omega}_\gamma - \vec{\omega}_{\rm b}), \tag{2.41}$$

$$\partial_\tau \vec{B} = \vec{\nabla}\times(\vec{v}_{\rm b}\times\vec{B}) + \frac{\nabla^2\vec{B}}{4\pi\sigma} + \frac{m_{\rm i} a}{e\,R_{\rm b}}\epsilon'(\vec{\omega}_{\rm b} - \vec{\omega}_\gamma), \tag{2.42}$$

$$\partial_\tau \vec{\omega}_\gamma = \epsilon'(\vec{\omega}_{\rm b} - \vec{\omega}_\gamma), \tag{2.43}$$

where $\epsilon' = a\Gamma_{{\rm e}\gamma}$ is the differential optical depth where, as usual, the contribution of the ions has been neglected. In the tight coupling limit Eqs. (2.41), (2.42) and (2.43) imply that $\vec{\omega}_{{\rm b}\gamma} \simeq \vec{\omega}_{\rm b} \simeq \vec{\omega}_\gamma$ while $\vec{\omega}_{{\rm b}\gamma}$ obeys

$$\partial_\tau \vec{\omega}_{{\rm b}\gamma} + \frac{\mathcal{H}R_{\rm b}}{R_{\rm b}+1}\vec{\omega}_{{\rm b}\gamma} = R_{\rm b}\frac{\vec{\nabla}\times(\vec{J}\times\vec{B})}{\rho_{\rm b}\,a^4(R_{\rm b}+1)}. \tag{2.44}$$

In analogy with what has been done before, the conservation laws can be derived by combining Eqs. (2.41) and (2.42)

$$\partial_\tau\left(\vec{B} + \frac{m_{\rm i}}{e}\,a\,\vec{\omega}_{\rm b}\right) = \vec{\nabla}\times(\vec{v}_{\rm b}\times\vec{B}) + \frac{\nabla^2\vec{B}}{4\pi\sigma} + \frac{m_{\rm i}}{e}\frac{\vec{\nabla}\times(\vec{J}\times\vec{B})}{a^3\rho_{\rm b}}. \tag{2.45}$$

From Eqs. (2.42) and (2.43) and by neglecting the spatial gradients it also follows

$$\partial_{\tau}\left(\vec{B} - \frac{a}{R_{\rm b}}\frac{m_{\rm i}}{e}\vec{\omega}_{\gamma}\right) = 0. \tag{2.46}$$

Equations (2.45) and (2.46) are separately valid, but, taken together and in the limit of tight baryon-photon coupling, they imply that the magnetic filed must be zero when the tight-coupling is exact (i.e. $\vec{\omega}_{\gamma} = \vec{\omega}_{\rm b}$). In spite of the various physical regimes encountered in the analysis of the evolution of the vorticity the key point is to find a suitable source of large-scale vorticity which could be converted, in some way into a large-scale magnetic field [28] (see also [29, 30]). The conversion can not only occur prior to matter-radiation equality but also after [5] in the regime where, as explained, the baryon-photon coupling becomes weak. Indeed, Eqs. (2.32) and (2.45) have the same dynamical content when the spatial gradients are neglected and the only difference involves the coupling to the photons.

There have been, through the years, suggestions involving primordial turbulence (see the interesting accounts of Refs. [31]), cosmic strings with small scale structure (see, e. g. [32, 33, 34]). Since matter flow in baryonic wakes is turbulent, velocity gradients will be induced in the flow by the small-scale wiggles of the string producing ultimately the vorticity. Dynamical friction between cosmic strings and matter may provide a further source of vorticity [33]. There have been also studies trying to generate large-scale magnetic fields in the context of superconducting cosmic strings (see, for instance,[34] and references therein). The possible generation of large-scale magnetic fields prior to hydrogen recombination has been discussed in [35, 36, 37] (see also [38]). The vorticity required in order to produce the magnetic fields is generated, according to [35], by the photon diffusion at second order in the temperature fluctuations. In a similar perspective Hogan [37] got less optimistic estimates which, according to [35, 36], should be attributed to different approximation schemes employed in the analysis. Along this perspective various analyses discussed higher-order effects using the conventional perturbative expansion in the presence of the relativistic fluctuations of the geometry [39]. In the present paper, as already mentioned, we are going to follow a different route since we intend to use the gradient expansion for a direct estimate of the vorticity.

3 Vorticity evolution in gradient expansion

The conservation laws derived in section 2 hold under the hypothesis that the spatial gradients are neglected in the evolution equations of the vorticity. The logic of the gradient expansion [17, 18, 19, 20, 21, 22] can be combined with the tenets of the drift approximation [23, 24, 25] in the context of the ADM decomposition [15, 16]. It will be shown hereunder that the resulting formalism [14] provides a more general description of the angular momentum transfer between the various species of the plasma. Consider therefore the standard ADM decomposition where the shift vectors are set to zero but the lapse function kept arbitrary,

i.e. $g_{00}(\vec{x},\tau) = N^2(\vec{x},\tau)$ and $g_{ij}(\vec{x},\tau) = -\gamma_{ij}(\vec{x},\tau)$. In this case the Maxwell equations can be written as

$$\vec{\partial}\cdot\vec{E} = 4\pi e[n_{\mathrm{i}} - n_{\mathrm{e}}], \qquad \vec{\partial}\cdot\vec{B} = 0, \tag{3.1}$$

$$\partial_\tau\vec{B} + \vec{\partial}\times\vec{E} = 0, \qquad \vec{\partial}\times\vec{B} = 4\pi e\Big[n_{\mathrm{i}}\,\vec{v}_{\mathrm{i}} - n_{\mathrm{e}}\,\vec{v}_{\mathrm{e}}\Big] + \partial_\tau\vec{E}, \tag{3.2}$$

where the rescaled electric and magnetic fields are given by:

$$E^i(\vec{x},\tau) = \left(\frac{\sqrt{\gamma}}{N}\right)_{(\vec{x},\tau)}\mathcal{E}^i(\vec{x},\tau), \qquad B^i(\vec{x},\tau) = \left(\frac{\sqrt{\gamma}}{N}\right)_{(\vec{x},\tau)}\mathcal{B}^i(\vec{x},\tau); \tag{3.3}$$

in Eq. (3.3) the subscripts specify that the rescaling is space-time dependent. The rescaled concentrations are

$$n_{\mathrm{i}}(\vec{x},\tau) = \sqrt{\gamma}\,\tilde{n}_{\mathrm{i}}(\vec{x},\tau), \qquad n_{\mathrm{e}}(\vec{x},\tau) = \sqrt{\gamma}\,\tilde{n}_{\mathrm{e}}(\vec{x},\tau). \tag{3.4}$$

The shorthand notation[10] employed in Eqs. (3.1)–(3.3) implies for a generic vector A^i,

$$\vec{\partial}\cdot\vec{A} \equiv \partial_i A^i, \qquad (\vec{\partial}\times\vec{A})^i = \partial_j\Big[N\gamma^{ik}\,\gamma^{jn}\,\eta_{nmk}\,A^m\Big]. \tag{3.5}$$

In appendix A some relevant complements on this formalism have been collected to avoid a digression from the main line of arguments contained in the present section. Two relevant aspects must anyway be borne in mind:

- in the conformally flat limit (i.e. $N(\vec{x},\tau) \to a(\tau)$ and $\gamma_{ij}(\vec{x},\tau) \to a^2(\tau)\delta_{ij}$) Eqs. (3.1) and (3.2) reproduce exactly Eqs. (2.2) and (2.3);
- the same comment holds for all the other fields (i.e. comoving or physical) involved in the fully inhomogeneous description.

Using the generalized curl operator of Eq. (3.5) the vorticity of the ions, of the electrons and of the photons can be written as

$$\omega_{\mathrm{i}}^i = \partial_j(\Lambda_m^{ij}\,v_{\mathrm{i}}^m), \qquad \omega_{\mathrm{e}}^i = \partial_j(\Lambda_m^{ij}\,v_{\mathrm{e}}^m), \qquad \omega_\gamma^i = \partial_j(\Lambda_m^{ij}\,v_\gamma^m), \tag{3.6}$$

where K_{ij} is the extrinsic curvature (see appendix A) while Λ_m^{ij} and $\overline{\Lambda}_m^{ij}$ are defined as[11]:

$$\Lambda_m^{ij} = N\gamma^{ik}\,\gamma^{jn}\,\eta_{nmk}, \qquad \overline{\Lambda}_m^{ij} = 2N^2[K^{ik}\,\gamma^{jn} + K^{jn}\,\gamma^{ik}]\eta_{nmk}. \tag{3.7}$$

[10] Note that the operators introduced in Eqs. (3.1)–(3.3) are the generalized curl, divergence and gradient operators; they reduce to the conventional curl, divergence and gradient operators in the conformally flat limit.

[11] Recall that $\eta_{abc} = \sqrt{\gamma}\,\epsilon_{abc}$ and that $\eta^{abc} = \epsilon^{abc}/\sqrt{\gamma}$.

Using Eqs. (3.6)–(3.7) as well as the evolution equations of the velocities (see, Eqs. (A.8)–(A.9)), the evolution for the vorticity of the electrons and of the ions can be written, respectively, as[12]

$$\begin{aligned}
&\partial_{\tau}\omega_{\mathrm{e}}^{i} + \biggl(NK - \frac{\partial_{\tau}N}{N}\biggr)\omega_{\mathrm{e}}^{i} - \mathcal{G}_{k}^{i}\omega_{\mathrm{e}}^{k} - \mathcal{F}_{\mathrm{e}}^{i} = \\
&-\frac{e\tilde{n}_{\mathrm{e}}N^{2}}{\rho_{\mathrm{e}}\sqrt{\gamma}}\Bigl\{(\vec{\partial}\times\vec{E})^{i} + [\vec{\partial}\times(\vec{v}_{\mathrm{e}}\times\vec{B})]^{i}\Bigr\} + N\Gamma_{\mathrm{ei}}(\omega_{\mathrm{i}}^{i} - \omega_{\mathrm{e}}^{i}) + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{e}}}N\Gamma_{\mathrm{e}\gamma}(\omega_{\gamma}^{i} - \omega_{\mathrm{e}}^{i}), \quad (3.8)\\
&\partial_{\tau}\omega_{\mathrm{i}}^{i} + \biggl(NK - \frac{\partial_{\tau}N}{N}\biggr)\omega_{\mathrm{i}}^{i} - \mathcal{G}_{k}^{i}\omega_{\mathrm{i}}^{k} - \mathcal{F}_{\mathrm{i}}^{i} = \\
&\frac{e\tilde{n}_{\mathrm{i}}N^{2}}{\rho_{\mathrm{i}}\sqrt{\gamma}}\Bigl\{(\vec{\partial}\times\vec{E})^{i} + [\vec{\partial}\times(\vec{v}_{\mathrm{i}}\times\vec{B})]^{i}\Bigr\} + N\Gamma_{\mathrm{ie}}\frac{\rho_{\mathrm{e}}}{\rho_{\mathrm{i}}}(\omega_{\mathrm{e}}^{i} - \omega_{\mathrm{i}}^{i}) + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{i}}}N\Gamma_{\mathrm{i}\gamma}(\omega_{\gamma}^{i} - \omega_{\mathrm{i}}^{i}). \quad (3.9)
\end{aligned}$$

Similarly, from Eq. (A.10) the evolution equation for the vorticity of the photons can be written as

$$\partial_{\tau}\omega_{\gamma}^{i} + \biggl[\frac{4}{3}NK - \frac{\partial_{\tau}N}{N}\biggr]\omega_{\gamma}^{i} - \mathcal{G}_{k}^{i}\omega_{\gamma}^{k} - \mathcal{F}_{\gamma}^{i} = N\Gamma_{\gamma\mathrm{e}}(\omega_{\mathrm{e}}^{i} - \omega_{\gamma}^{i}) + N\Gamma_{\gamma\mathrm{i}}(\omega_{\mathrm{i}}^{i} - \omega_{\gamma}^{i}). \tag{3.10}$$

The quantities $\mathcal{F}_{\mathrm{e}}^{i}$, $\mathcal{F}_{\mathrm{i}}^{i}$ and $\mathcal{F}_{\gamma}^{i}$ appearing in Eqs. (3.8), (3.9) and (3.10) are of the same order of the other terms appearing in the equations and they are defined as

$$\begin{aligned}
\mathcal{F}_{\mathrm{e}}^{i} &= \partial_{j}\Bigl(\overline{\Lambda}_{m}^{ij}v_{\mathrm{e}}^{m}\Bigr) + \frac{4}{3}N\Gamma_{\gamma\mathrm{e}}\partial_{j}\Bigl(\frac{\rho_{\gamma}}{\rho_{\mathrm{e}}}\Bigr)\Lambda_{m}^{ij}(v_{\gamma}^{m} - v_{\mathrm{e}}^{m}),\\
&+ \partial_{j}\mathcal{G}_{a}^{m}\Lambda_{m}^{ij}v_{\mathrm{e}}^{a} - N\partial_{j}K\Lambda_{m}^{ij}v_{\mathrm{e}}^{m} - \partial_{j}\Bigl(\frac{e\tilde{n}_{\mathrm{e}}N^{2}}{\rho_{\mathrm{e}}\sqrt{\gamma}}\Bigr)\Lambda_{m}^{ij}\Bigl[E^{m} + (\vec{v}_{\mathrm{e}}\times\vec{B})^{m}\Bigr], \quad (3.11)\\
\mathcal{F}_{\mathrm{i}}^{i} &= \partial_{j}\Bigl(\overline{\Lambda}_{m}^{ij}v_{\mathrm{i}}^{m}\Bigr) + \frac{4}{3}N\Gamma_{\gamma\mathrm{i}}\partial_{j}\Bigl(\frac{\rho_{\gamma}}{\rho_{\mathrm{i}}}\Bigr)\Lambda_{m}^{ij}(v_{\gamma}^{m} - v_{\mathrm{i}}^{m}) + N\partial_{j}\Bigl(\frac{\rho_{\mathrm{e}}}{\rho_{\mathrm{i}}}\Bigr)\Lambda_{m}^{ij}\Gamma_{\mathrm{ie}}(v_{\mathrm{e}}^{m} - v_{\mathrm{i}}^{m}),\\
&+ \partial_{j}\mathcal{G}_{a}^{m}\Lambda_{m}^{ij}v_{\mathrm{i}}^{a} - N\partial_{j}K\Lambda_{m}^{ij}v_{\mathrm{i}}^{m} + \partial_{j}\Bigl(\frac{e\tilde{n}_{\mathrm{i}}N^{2}}{\rho_{\mathrm{i}}\sqrt{\gamma}}\Bigr)\Lambda_{m}^{ij}\Bigl[E^{m} + (\vec{v}_{\mathrm{i}}\times\vec{B})^{m}\Bigr], \quad (3.12)\\
\mathcal{F}_{\gamma}^{i} &= \partial_{j}\Bigl(\overline{\Lambda}_{k}^{ij}\,v_{\gamma}^{k}\Bigr) + \Lambda_{k}^{ij}\,v_{\gamma}^{q}\partial_{j}\mathcal{G}_{q}^{k} - \frac{4}{3}N\partial_{j}K\,\Lambda_{k}^{ij}\,v_{\gamma}^{k} - \frac{N^{2}}{4}\partial_{j}\Bigl\{\frac{\Lambda_{k}^{ij}}{\rho_{\gamma}}\partial_{m}\Bigl[\rho_{\gamma}\gamma^{mk}\Bigr]\Bigr\}. \quad (3.13)
\end{aligned}$$

The generalized scalar and vector products appearing in Eqs. (3.11), (3.12) and (3.13) are defined as

$$\vec{F}\cdot\vec{G} = \gamma_{mn}F^{m}G^{n}, \qquad (\vec{F}\times\vec{G})^{k} = \frac{\gamma_{in}\gamma_{m\ell}}{N}F^{n}G^{m}\eta^{i\ell k}, \tag{3.14}$$

and coincide with the ordinary scalar and vector products in the conformally flat limit introduced after Eq. (3.5). The velocity fields appearing in Eqs. (3.8) and (3.9) are all subjected to the fully inhomogeneous form of the momentum constraint implying, from Eq. (A.6),

$$\frac{1}{N}\Bigl(\nabla_{i}K - \nabla_{k}K_{i}^{k}\Bigr) = \ell_{\mathrm{P}}^{2}(p+\rho)u^{0}u_{i}, \qquad u^{0} = \frac{1}{N}\sqrt{1+u^{2}}, \tag{3.15}$$

[12] We shall focus, without loss of generality, on the situation where the lapse function is homogeneous, i.e. $N(\vec{x},\tau) = N(\tau)$; in this case the already lengthy expressions will be more manageable since the spatial derivatives of the lapse function will vanish.

where $u^2 = u^i u^j \gamma_{ij}$ and where u^0 and u^i can also be defined in terms of the total velocity field v^i which turns out to be the weighted sum of the velocity fields of the electrically charged and of the electrically neutral species, i.e.

$$(p+\rho)v^k = \sum_a (p_{\rm a} + \rho_{\rm a}) v_{\rm a}^k = \rho_{\rm e} v_{\rm e}^k + \rho_{\rm i} v_{\rm i}^k + \frac{4}{3}\rho_\gamma v_\gamma^k + \frac{4}{3}\rho_\nu v_\nu^k + \rho_{\rm c} v_{\rm c}^k, \tag{3.16}$$

where the contribution of the cold dark matter particles and of the massless neutrinos has been also added. The explicit connection between u^0, u^i and v^i is given by:

$$u^0 = \frac{\cosh y}{N}, \qquad u^i = \frac{v^i}{N}\cosh y, \qquad \cosh y = \frac{1}{\sqrt{1 - v^2/N^2}}, \tag{3.17}$$

where $v^2 = v^i v^j \gamma_{ij}$. In terms of v^i and v^2 the momentum constraint of Eq. (3.15) can also be written as

$$\ell_{\rm P}^2 (p+\rho)\frac{v^i}{N} = \biggl(1 - \frac{v^2}{N^2}\biggr)\nabla_k \biggl(K^{ki} - K\gamma^{ki}\biggr). \tag{3.18}$$

All the discussion of section 2 can be generalized to the fully inhomogeneous case and we shall be particularly interested in the generalization of the conservation laws determining the angular momentum exchange between the various species. Consider then the situation where the electron-photon rate dominates against the Coulomb rate. In this case the fully inhomogeneous form of the Ohm law reads

$$-E^k - (\vec{v}_{\rm e}\times\vec{B})^k + \frac{J^k}{\sigma} + \frac{4}{3\,e}\frac{\rho_\gamma}{\rho_{\rm b}} m_{\rm i}\Gamma_{{\rm e}\gamma}\frac{\sqrt{\gamma}}{N}(v_\gamma^k - v_{\rm e}^k) = 0. \tag{3.19}$$

By taking the generalized curl of Eq. (3.19) (see Eq. (3.5)) the following equation can be obtained

$$\begin{aligned} &-\vec{\partial}\times\vec{E} - \vec{\partial}\times(\vec{v}_{\rm e}\times\vec{B}) + \vec{\partial}\times(\vec{J}/\sigma) \\ &+\frac{4}{3\,e}\frac{\rho_\gamma}{\rho_{\rm b}} m_{\rm i}\Gamma_{{\rm e}\gamma}\frac{\sqrt{\gamma}}{N}(\vec{\omega}_\gamma - \vec{\omega}_{\rm e}) - \frac{4}{3}\frac{m_{\rm i}}{e}N^2(\vec{v}_\gamma - \vec{v}_{\rm e})\times\vec{\partial}\biggl[\Gamma_{{\rm e}\gamma}\frac{\sqrt{\gamma}}{N}\frac{\rho_\gamma}{\rho_{\rm b}}\biggr] = 0, \end{aligned} \tag{3.20}$$

where, consistently with Eq. (3.14), the last term at the left hand side is defined in terms of the generalized vector product and it vanishes exactly in the conformally flat limit. By assuming, as physically plausible prior to decoupling, that the conductivity is homogeneous, Eqs. (3.1) and (3.2) can be used inside Eq. (3.20) and the final equation will then be:

$$\begin{aligned} \partial_\tau \vec{B} &= \vec{\partial}\times(\vec{v}_{\rm e}\times\vec{B}) - \frac{1}{4\pi\sigma}\vec{\partial}\times(\vec{\partial}\times\vec{B}) - \frac{4}{3\,e}\frac{\rho_\gamma}{\rho_{\rm b}} m_{\rm i}\Gamma_{{\rm e}\gamma}\frac{\sqrt{\gamma}}{N}(\vec{\omega}_\gamma - \vec{\omega}_{\rm e}) \\ &+\frac{4}{3}N^2\frac{m_{\rm i}}{e}(\vec{v}_\gamma - \vec{v}_{\rm e})\times\vec{\partial}\biggl[\Gamma_{{\rm e}\gamma}\frac{\sqrt{\gamma}}{N}\frac{\rho_\gamma}{\rho_{\rm b}}\biggr]. \end{aligned} \tag{3.21}$$

Equation (3.21) reduces, in the conformally flat limit, to Eq. (2.29). The same logic can be applied in all the other derivations and the obtained result expanded to first order in

the spatial gradients with the result that the generalized system for the evolution of the vorticities reads

$$\partial_\tau \omega_{\rm i}^{k} = \biggl(N K + 2 \frac{\partial_{\tau} N}{N}\biggr)\omega_{\rm i}^{k} - \frac{e \tilde{n}_{\rm i}}{\rho_{\rm i} \sqrt{\gamma}}\, N^2\, \partial_{\tau} B^{k}, \tag{3.22}$$

$$\partial_{\tau} B^{k} = - \frac{4}{3\, e} \Gamma_{{\rm e}\gamma} \frac{\rho_{\gamma}}{\rho_{\rm b}} m_{\rm i} \frac{\sqrt{\gamma}}{N} (\omega_{\gamma}^{k} - \omega_{\rm e}^{k}), \tag{3.23}$$

$$\partial_{\tau} \omega_{\gamma}^{k} = \biggl(\frac{2}{3} N K + 2 \frac{\partial_{\tau} N}{N}\biggr) \omega_{\gamma}^{k} + N \Gamma_{{\rm e}\gamma} (\omega_{\rm e}^{k} - \omega_{\gamma}^{k}). \tag{3.24}$$

Equations (3.22), (3.23) and (3.24) reduce, respectively, to Eqs. (2.29), (2.30) and (2.31) in the conformally flat limit. Equations (3.22), (3.23) and (3.24) apply in the situation where the magnetic fields are initially zero and do not contribute to the extrinsic curvature so that $K_{i}^{j} = K/3 \delta_{i}^{j} + \overline{K}_{i}^{j}$ with $\overline{K}_{i}^{j} =0$. In this case Eqs. (3.22)–(3.24) reduce to a pair of remarkable conservation laws whose explicit expression, up to spatial gradients, is

$$\partial_{\tau}\biggl[\frac{\sqrt{\gamma}}{N^2} \omega_{\rm i}^{k} + \frac{e \tilde{n}_{\rm i}}{\rho_{\rm b}} B^{k} \biggr] =0, \tag{3.25}$$

$$\partial_{\tau} \biggl[B^{k} - \frac{m_{\rm i}}{e\, \overline{R}_{\rm b}} \frac{\gamma^{1/3}}{N^2} \omega_{\gamma}^{k}\biggr] = 0, \tag{3.26}$$

where $\overline{R}_{\rm b}(\vec{x},\tau_1)$ is a constant in time (but not in space) and come from the inhomogeneous generalization of $R_{\rm b}(\vec{x},\tau)$:

$$R_{\rm b}(\vec{x},\tau) = \frac{3}{4} \frac{\rho_{\rm b}(\vec{x},\tau)}{\rho_{\gamma}(\vec{x},\tau)} = \overline{R}_{\rm b}(\vec{x},\tau_1) \gamma^{1/6}. \tag{3.27}$$

The evolution of the vorticity of the baryons as well as the tight coupling between the baryons and the photons can be discussed in full analogy with the considerations already developed above in the case of the electron-photon coupling. The inhomogeneous generalization of the Ohm law when the Coulomb scattering dominates against both the electron-photon and the ion-photon coupling has been derived in Ref. [14] (see Eq. (3.34)). To leading order in the gradient expansion the evolution of the baryon vorticity can be written as

$$\partial_{\tau} \omega_{\rm b}^{k} = \biggl(N K + 2 \frac{\partial_{\tau} N}{N} \biggr) \omega_{\rm b}^{k} + \frac{\epsilon'}{R_{b}} (\omega_{\gamma}^{k} - \omega_{\rm b}^{k}), \tag{3.28}$$

$$\partial_{\tau} B^{k} = - \frac{m_{\rm i}}{e} \frac{\epsilon'}{R_{\rm b}} \frac{\sqrt{\gamma}}{N^2} (\omega_{\gamma}^{k} - \omega_{\rm b}^{k}). \tag{3.29}$$

where $\epsilon' = N \Gamma_{{\rm e}\gamma}$ is the inhomogeneous generalization of the optical depth. By eliminating ϵ' between Eqs. (3.28) and (3.29) the following equation

$$\partial_{\tau}\biggl[\frac{\sqrt{\gamma}}{N^2} \omega_{\rm b}^{k} + \frac{e \tilde{n}_{\rm i}}{\rho_{\rm b}} B^{k} \biggr] =0 \tag{3.30}$$

is readily obtained. Note that Eq. (3.30) coincides, up to spatial gradients and in the conformally flat limit, with Eq. (2.45).

4 Maximal vorticity induced by the geometry

In this paper the the expansion is organized not in terms of the relative magnitude of the gravitational and electromagnetic fluctuations but in terms of the number of gradients carried by each order of the expansion. From the momentum constraint (see Eq. (3.18)), the total velocity field can be written, formally,

$$v^{i} = -\frac{N\,S^{i}}{2S^{2}}\Big[1-\sqrt{1+4S^{2}}\Big] \simeq NS^{i}\Big[1-S^{2}+\mathcal{O}(\epsilon^{3})\Big]+\mathcal{O}(\epsilon^{4}), \tag{4.1}$$

$$S^{i} = \frac{1}{\ell_{\mathrm{P}}^{2}(p+\rho)}\nabla_{k}\Big(K^{ki}-K\gamma^{ki}\Big), \tag{4.2}$$

where the orders of the expansion appearing in Eq. (4.1) are defined by the number of gradients. From Eqs. (4.1) and (3.6)–(3.7) the total vorticity can be written as

$$\omega_{\mathrm{tot}}^{i} = \partial_{j}\Big\{N\Lambda^{ij}_{m}S^{m}\Big[1-S^{2}+\mathcal{O}(\epsilon^{3})\Big]\Big\}. \tag{4.3}$$

To implement the gradient expansion let us parametrize the geometry as

$$\gamma_{ij}(\vec{x},\tau) = a^{2}(\tau)[\alpha_{ij}(\vec{x})+\beta_{ij}(\vec{x},\tau)], \qquad \gamma^{ij}(\vec{x},\tau) = \frac{1}{a^{2}(\tau)}[\alpha^{ij}(\vec{x})-\beta^{ij}(\vec{x},\tau)]. \tag{4.4}$$

and keep the lapse function homogeneous, i.e. $N(\tau) = a(\tau)$; $\alpha_{ij}(\vec{x})$ does not contain any spatial gradient while $\beta_{ij}(\vec{x},\tau)$ contains at least one spatial gradient. The extrinsic curvature becomes:

$$K_{i}^{j} = -\Big(\frac{\mathcal{H}}{a}\delta_{i}^{j}+\frac{1}{2}\frac{\partial_{\tau}\beta_{i}^{j}}{a}\Big), \qquad K^{ik} = -\frac{1}{a^{3}}\Big[\mathcal{H}(\alpha^{ik}-\beta^{ik})+\frac{1}{2}\partial_{\tau}\beta^{ik}\Big]. \tag{4.5}$$

Furthermore we have also that the spatial Christoffel are:

$$\Gamma_{ka}^{k} = \partial_{a}\ln\sqrt{\gamma} = \frac{1}{2}\Big(\frac{\partial_{a}\alpha}{\alpha}+\partial_{a}\beta\Big), \tag{4.6}$$

$$\Gamma_{ab}^{m} = \frac{1}{2}\Big[\alpha^{mn}\lambda_{nab}+\alpha^{mn}\overline{\lambda}_{nab}-\beta^{mn}\lambda_{nab}\Big], \tag{4.7}$$

where λ_{nab} and $\overline{\lambda}_{nab}$

$$\lambda_{nab} = -\partial_{n}\alpha_{ab}+\partial_{b}\alpha_{na}+\partial_{a}\alpha_{bn}, \tag{4.8}$$

$$\overline{\lambda}_{nab} = -\partial_{n}\beta_{ab}+\partial_{b}\beta_{na}+\partial_{a}\beta_{bn}. \tag{4.9}$$

The relevant term appearing in the momentum constraint becomes then

$$\nabla_{k}\Big(K^{km}-K\gamma^{km}\Big) = \nabla_{k}K^{km}+\frac{\alpha^{km}\partial_{\tau}\partial_{k}\beta}{2a^{3}}, \tag{4.10}$$

where

$$
\begin{aligned}
\nabla_k K^{km} &= -\frac{1}{a^3}\Big[\frac{\mathcal{H}}{2\alpha}(\partial_a\alpha)\alpha^{am} + \mathcal{H}\partial_k\alpha^{km} + \mathcal{H}\frac{\alpha^{mi}\alpha^{ab}}{2}\lambda_{iab}\Big] \\
&+ \frac{1}{2a^3}\Big[-\partial_k\partial_\tau\beta^{km} - \frac{1}{2}\Big(\partial_\tau\beta^{ab}\alpha^{mi}\Big)\lambda_{iab} - \Big(\frac{\partial_a\alpha}{2\alpha}\Big)\partial_\tau\beta^{am} \\
&- \mathcal{H}\partial_a\beta\alpha^{am} + \mathcal{H}\Big(\frac{\partial_a\alpha}{\alpha}\Big)\beta^{am} + 2\mathcal{H}\partial_k\beta^{km} - \mathcal{H}\alpha^{mi}\alpha^{ab}\overline{\lambda}_{iab} \\
&+ \mathcal{H}\Big(\alpha^{mi}\beta^{ab} + \alpha^{ab}\beta^{mi}\Big)\lambda_{iab}\Big]. \qquad (4.11)
\end{aligned}
$$

Equation (4.10) can therefore be written as

$$
\begin{aligned}
\nabla_k\Big(K^{km} - K\gamma^{km}\Big) &= -\frac{\mathcal{H}}{a^3}\Big[\frac{1}{2\alpha}(\partial_a\alpha)\alpha^{am} + \partial_k\alpha^{km} + \frac{\alpha^{mi}\alpha^{ab}}{2}\lambda_{iab}\Big] \\
&+ \frac{1}{2a^3}\Big\{\alpha^{km}\partial_k\partial_\tau\beta - \partial_k\partial_\tau\beta^{km} \\
&- \Big[\frac{1}{2}\Big(\partial_\tau\beta^{ab}\alpha^{mi}\Big)\lambda_{iab} + \Big(\frac{\partial_a\alpha}{2\alpha}\Big)\partial_\tau\beta^{am}\Big] \\
&+ \mathcal{H}\Big[2\partial_k\beta^{km} - \partial_a\beta\alpha^{am} + \Big(\frac{\partial_a\alpha}{\alpha}\Big)\beta^{am} - \alpha^{mi}\alpha^{ab}\overline{\lambda}_{iab} \\
&+ \Big(\alpha^{mi}\beta^{ab} + \alpha^{ab}\beta^{mi}\Big)\lambda_{iab}\Big]\Big\}. \qquad (4.12)
\end{aligned}
$$

The previous expression can also be recast in a more handy form:

$$
\nabla_k\Big(K^{km} - K\gamma^{km}\Big) = -\frac{\mathcal{H}}{a^3}\mathcal{Z}^m(\alpha) + \frac{1}{2a^3}\Big[\mathcal{I}_1^m(\alpha,\beta) - \mathcal{I}_2^m(\alpha,\beta) + \mathcal{H}\mathcal{I}_3^m(\alpha,\beta)\Big], \qquad (4.13)
$$

where the three functionals of $\alpha_{ij}(\vec{x})$ and $\beta_{ij}(\vec{x},\tau)$ are defined as

$$
\begin{aligned}
\mathcal{Z}^m(\alpha) &= \frac{1}{2}\frac{\partial_a\alpha}{\alpha}\alpha^{am} + \partial_q\alpha^{qm} + \frac{\alpha^{mq}\alpha^{ab}}{2}\lambda_{qab}, \qquad (4.14) \\
\mathcal{I}_1^m(\alpha,\beta) &= \alpha^{qm}\partial_q\partial_\tau\beta - \partial_\tau\partial_q\beta^{qm}, \qquad (4.15) \\
\mathcal{I}_2^m(\alpha,\beta) &= \frac{\alpha^{qm}}{2}(\partial_\tau\beta^{ab})\lambda_{qab} + \frac{\partial_a\alpha}{2\alpha}\partial_\tau\beta^{am}, \qquad (4.16) \\
\mathcal{I}_3^m(\alpha,\beta) &= 2\partial_q\beta^{qm} - (\partial_a\beta)\alpha^{am} + \frac{\partial_a\alpha}{\alpha}\beta^{am} + \lambda_{qab}\Big(\alpha^{qm}\beta^{ab} + \alpha^{ab}\beta^{mq}\Big) \\
&- \alpha^{mq}\alpha^{ab}\overline{\lambda}_{qab}. \qquad (4.17)
\end{aligned}
$$

With the result of Eq. (4.13) we can compute the first relevant part of the final expression, namely:

$$
\begin{aligned}
N^2\gamma^{aj}\gamma^{in}\eta_{amn}\nabla_k\Big(K^{km} - K\gamma^{km}\Big) &= \frac{\sqrt{\alpha}}{a^2}\Big(1+\frac{\beta}{2}\Big)\Big\{-\mathcal{H}\alpha^{kj}\alpha^{in}\mathcal{Z}^m(\alpha)\epsilon_{kmn} \\
+ \mathcal{H}\Big(\alpha^{kj}\beta^{in} + \alpha^{in}\beta^{kj}\Big)\mathcal{Z}^m(\alpha)\epsilon_{kmn} &+ \frac{\alpha^{kj}\alpha^{in}}{2}\epsilon_{kmn}\Big[\mathcal{I}_1^m(\alpha,\beta) - \mathcal{I}_2^m(\alpha,\beta) \\
+ \mathcal{H}\mathcal{I}_3^m(\alpha,\beta)\Big]\Big\}. \qquad (4.18)
\end{aligned}
$$

Recalling that, furthermore[13]

$$\ell_{\rm P}^2 (p+\rho) a^2 = \frac{3 \mathcal{H}_1^2 (1 + w)}{\alpha^{(w+1)/2}\,(1 + \beta/2)^{w+1}} \biggl(\frac{a_1}{a}\biggr)^{3 w +1}, \qquad \ell_{\rm P}^2 \overline{\rho}_{1} a_{1}^2 = 3 \mathcal{H}_{1}^2. \tag{4.19}$$

Putting all the various parts of the calculation together we have that, from Eq. (4.3),

$$\omega^{i}_{\rm tot} = \partial_{j} {\mathcal A}^{ij}, \qquad {\mathcal A}^{ij} = \frac{N^2\,\gamma^{kj}\,\gamma^{in}\,\eta_{kmn}}{\ell_{\rm P}^2 (p + \rho)} \nabla_{a}\biggl(K^{am} - \gamma^{am} K\biggr), \tag{4.20}$$

then the quantity ${\mathcal A}^{ij}$ becomes:

$$\begin{aligned} {\mathcal A}^{ij}(\alpha,\beta) &= \frac{\alpha^{(w+2)/2}}{3 \mathcal{H}_{1}^2 (w + 1)} \biggl(\frac{a}{a_1}\biggr)^{3 w+1} \biggl\{ - \mathcal{H} \alpha^{kj} \alpha^{in} {\mathcal Z}^{m}(\alpha) \epsilon_{kmn} \\ &+ \mathcal{H}\biggl[\alpha^{kj} \beta^{in} + \alpha^{in} \beta^{kj} \biggr] {\mathcal Z}^{m}(\alpha) \epsilon_{kmn} + \frac{\alpha^{kj}\alpha^{in}}{2} \epsilon_{kmn} \biggl[{\mathcal I}^{m}_{1}(\alpha,\beta) \\ &- {\mathcal I}^{m}_{2}(\alpha,\beta) + \mathcal{H} {\mathcal I}^{m}_{3}(\alpha,\beta)\biggr] - \frac{\mathcal{H}}{2} (w + 2) \beta \alpha^{kj} \alpha^{in} {\mathcal Z}^{m}(\alpha) \epsilon_{kmn}\biggr\}. \end{aligned} \tag{4.21}$$

The first line at the right hand side of Eq. (4.21) does not contain any spatial gradient and it is therefore ${\mathcal O}(\alpha)$. The remaining part of the expression at the right hand side of the relation reported in Eq. (4.21) are instead ${\mathcal O}(\beta)$. Sticking to the situation treated in the present paper the explicit form of $\beta_{ij}(\vec{x},\tau)$ can be determined in terms of $\alpha_{ij}(\vec{x})$ by solving the remaining Einstein equations written in terms of the ADM decomposition [15, 16]. For this purpose Eqs. (A.5) and (A.7) can be written, respectively, as

$$\partial_{\tau} K - N {\rm Tr} K^2 = \frac{N \ell_{\rm P}^2}{2} (3 p + \rho), \tag{4.22}$$

$$\partial_{\tau} K_{i}^{j} - N K K_{i}^{j} - N r_{i}^{j} = \frac{N \ell_{\rm P}^2}{2} (p - \rho) \delta_{i}^{j}. \tag{4.23}$$

Using Eqs. (4.4) and (4.5) into Eqs. (4.22), the following pair of conditions are obtained

$$\partial_{\tau}\biggl(\frac{\partial_{\tau} \beta}{2 a}\biggr) + \frac{\mathcal{H}}{a} \partial_{\tau} \beta = - \frac{a \ell_{\rm P}^2}{2} (3 p^{(1)} + \rho^{(1)}), \tag{4.24}$$

$$\partial_{\tau} \mathcal{H} = - \frac{a^2 \ell_{\rm P}^2}{2} (\rho^{(0)} + 3 p^{(0)}). \tag{4.25}$$

To obtain Eqs. (4.24) and (4.25) the total pressure and the total energy density have been separated as:

$$p(\vec{x},\tau) = p^{(0)}(\tau) + p^{(1)}(\vec{x},\tau), \qquad \rho(\vec{x},\tau) = \rho^{(0)}(\tau) + \rho^{(1)}(\vec{x},\tau), \tag{4.26}$$

[13] We shall assume that w, the dominant barotropic index of the fluid sources, is constant.

where $p^{(1)}(\vec{x},\tau)$ and $\rho^{(1)}(\vec{x},\tau)$ vanish in the conformally flat limit. Using Eqs. (4.4) and (4.5) into Eqs. (4.23) two further equations are obtained and they are:

$$\partial_\tau\left(\frac{\partial_\tau \beta_i^j}{2a}\right) + \mathcal{H}\frac{\partial_\tau\beta}{2a}\delta_i^j + \frac{3\mathcal{H}}{2a}\partial_\tau\beta_i^j + a r_i^j = -\frac{a\ell_{\mathrm{P}}^2}{2}(p^{(1)} - \rho^{(1)})\delta_i^j, \tag{4.27}$$

$$\partial_\tau\mathcal{H} + 2\mathcal{H}^2 = -\frac{\ell_{\mathrm{P}}^2 a^2}{2}(p^{(0)} - \rho^{(0)}). \tag{4.28}$$

Solving Eqs. (4.25) and (4.28) under the hypothesis of constant barotropic index (already assumed in Eq. (4.19)), $p^{(1)}$ and $\rho^{(1)}$ can be eliminated between Eqs. (4.24) and (4.27) and it turns out that $\beta_{ij}(\vec{x},\tau)$ obeys the following evolution equation:

$$\partial_\tau^2\beta_i^j + 2\mathcal{H}\partial_\tau\beta_i^j + \delta_i^j\left(\frac{1-w}{1+3w}\partial_\tau^2\beta + 2\frac{1+w}{1+3w}\mathcal{H}\partial_\tau\beta\right) + 2a^2 r_i^j = 0. \tag{4.29}$$

By solving Eq. (4.29) the explicit form of β_{ij} can be written in a separable form as $\beta_i^j(\vec{x},\tau) = g(\tau)\mu_i^j(\vec{x})$ where:

$$g(\tau) = a^{3w+1}, \tag{4.30}$$

$$\mu_i^j(\vec{x}) = -\frac{4}{H_{\mathrm{i}}^2(3w+5)(3w+1)}\left[P_i^j(\vec{x}) + \frac{3w^2-6w-5}{4(9w+5)}P(\vec{x})\delta_i^j\right]. \tag{4.31}$$

Note that $P_i^j(\vec{x}) = r_i^j(\vec{x},\tau)a^2(\tau)$ accounts for the intrinsic curvature computed from $\alpha_{ij}(\vec{x})$. In Eqs. (4.22) and (4.23) the contribution of the velocity fields and of the magnetic fields has been neglected because they are subleading to $\mathcal{O}(\beta)$. In the following two sections we will therefore present the full estimate of the vorticity to first-order in the gradient expansion. If needed the first-order result, together with Eqs. (4.30) and (4.31) can be used to estimate the vorticity to higher order.

5 Vorticity to first-order in the gradient expansion

The simplest parametrization of $\alpha_{ij}(\vec{x})$ which does not contain spatial gradients can be written as

$$\alpha_{ij}(\vec{x}) = e^{-2\Psi(\vec{x})}\delta_{ij}, \qquad \alpha = \det\alpha_{ij} = e^{-6\Psi(\vec{x})}. \tag{5.1}$$

In this case it is easy to show that $\mathcal{Z}^m(\alpha) = 0$ and therefore the first-order in the gradient expansion vanishes identically. In the ΛCDM scenario the scalar mode appearing in Eq. (5.1) leads to a $|\Psi(\vec{x})| \ll 1$ and therefore, in practice, $\alpha_{ij}(\vec{x})$ is accurately estimated by $\delta_{ij} - 2\Psi(\vec{x})\delta_{ij}$. To have a $\mathcal{Z}^m(\alpha) \neq 0$ the contribution of the tensor modes must be included and $\alpha_{ij}(\vec{x})$ will then given by:

$$\alpha_{ij}(\vec{x}) = \left[\delta_{ij} + h_{ij}(\vec{x})\right], \qquad \alpha^{ij}(\vec{x}) = \left[\delta^{ij} - h^{ij} + h^{ik}h_k^j\right], \qquad \sqrt{\alpha} = \left[1 - \frac{1}{4}h_i^k\, h_k^i\right], \tag{5.2}$$

where h_{ij} is divergenceless and traceless, i.e. $\partial_i h^{ij} = h_i^i = 0$. It must be borne in mind that the scalar and the tensor modes, in the ΛCDM scenario and in its tensor extension, are defined in terms of the conventional perturbative expansion. As a consequence of the latter statement, the informations on the spatial inhomogeneities of the model are not specified by assigning the analog $\alpha_{ij}(\vec{x})$ (or $\gamma_{ij}(\vec{x},\tau)$ to a given order in the spatial gradients). On the contrary, as it is more natural, the scalar and tensor modes of the geometry are specified by assigning the corresponding power spectra at a given pivot scale. To evaluate the appropriate correlators defining the vorticity we shall need first to obtain the fluctuations in real space (as opposed to Fourier space). Therefore, as we will show in the present and in the following section, the idea will be first to compute the fluctuations in real space and then to use the obtained result for the determination of the correlators defining the vorticity. This procedure will circumvent the calculation of complicated convolutions and will also be perfectly suitable for the applications described in section 6. Using then Eq. (4.14) we have that

$$\mathcal{Z}^m(\alpha) = \partial_q \alpha^{qm} + \alpha^{mq}\alpha^{ab}\partial_b \alpha_{qa} = h^{qm}\, h^{ab}\, \partial_b h_{qa} + h^{ap}\, h_p^b\, \partial_b h_a^m. \tag{5.3}$$

From Eq. (4.21) the tensor $\mathcal{A}^{ij}(\alpha,\beta)$ can be computed to lowest order (i.e. by setting $\beta = 0$) and the result will therefore be written, using Eq. (5.3), as

$$\mathcal{A}^{ij}(\alpha) = -\frac{\mathcal{H}}{3\mathcal{H}_1^2(w+1)}\left(\frac{a}{a_1}\right)^{3w+1} \epsilon^{mij}\left[h^{a\ell}h_\ell^b \partial_b h_{am} + h_{mq}h^{ba}\partial_b h_a^q\right] + \mathcal{O}(\epsilon^2). \tag{5.4}$$

Finally, the total vorticity can be derived directly from Eq. (4.20)

$$\begin{aligned} \omega_{\mathrm{tot}}^i &= -\mathcal{L}(\tau,w)\,\epsilon^{mij}\partial_j\left[h^{a\ell}h_\ell^b\partial_b h_{am} + h_{mq}h^{ba}\partial_b h_{qa}\right] + \mathcal{O}(\epsilon^3), \\ \mathcal{L}(\tau,w) &= \frac{\mathcal{H}}{3\mathcal{H}_1^2(w+1)}\left(\frac{a}{a_1}\right)^{3w+1}. \end{aligned} \tag{5.5}$$

To give an explicit estimate of the primordial vorticity the relevant cosmological parameters will be taken to be the ones determined on the basis of the WMAP 7yr data alone [26, 27]. In the ΛCDM paradigm the sole source of curvature inhomogeneities is represented by the standard adiabatic mode whose associated power spectrum is assigned at the comoving pivot scale $k_{\mathrm{p}} = 0.002\,\mathrm{Mpc}^{-1}$ with characteristic amplitude $\mathcal{A}_{\mathcal{R}}$

$$\langle \mathcal{R}(\vec{k},\tau)\mathcal{R}(\vec{p},\tau)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k)\delta^{(3)}(\vec{k}+\vec{p}), \qquad \mathcal{P}_{\mathcal{R}}(k) = \mathcal{A}_{\mathcal{R}}\left(\frac{k}{k_{\mathrm{p}}}\right)^{n_{\mathrm{s}}-1}, \tag{5.6}$$

where n_{s} denotes the spectral index associated with the fluctuations of the spatial curvature. According to the WMAP 7yr data alone analyzed in the light of the ΛCDM paradigm and without tensors modes [26, 27] the determinations of $\mathcal{A}_{\mathcal{R}}$ and of n_{s} lead, respectively, to $\mathcal{A}_{\mathcal{R}} = (2.43 \pm 0.11) \times 10^{-10}$ and to $n_{\mathrm{s}} = 0.963 \pm 0.014$. The standard ΛCDM scenario, sometimes dubbed vanilla ΛCDM is defined by six pivotal parameters whose specific values

are, in the absence of tensor modes[14]

$$(\Omega_{\rm b},\,\Omega_{\rm c},\Omega_{\rm de},\,h_0,\,n_{\rm s},\,\epsilon_{\rm re}) \equiv (0.0449,\,0.222,\,0.734,\,0.710,\,0.963,\,0.088). \tag{5.7}$$

To estimate the correlation functions associated with Eqs. (5.4) and (5.5) it is mandatory to know in detail the numerical value of the correlation function of the tensor modes of the geometry which have not been detected so far but whose specific upper limits will determine the maximal magnetic field obtainable from the vorticity of the geometry. The tensor modes of the geometry are described in terms of a rotationally and parity invariant two-point function

$$\langle h_{ij}(\vec{x},\tau)\, h_{ij}(\vec{y},\tau)\rangle = \int \frac{dk}{k} \mathcal{P}_{\rm T}(k,\tau)\, \frac{\sin kr}{kr}, \tag{5.8}$$

where the tensor power spectrum at the generic time τ is given by the product of the appropriate transfer function multiplied by the primordial spectrum:

$$\mathcal{P}_{\rm T}(k,\tau) = \mathcal{M}(k,\,k_{\rm eq},\,\tau)\overline{\mathcal{P}}_{\rm T}(k), \qquad \overline{\mathcal{P}}_{\rm T}(k) = \mathcal{A}_{\rm T}\biggl(\frac{k}{k_{\rm p}}\biggr)^{n_{\rm T}}; \tag{5.9}$$

note that $\mathcal{A}_{\rm T}$ is the amplitude of the tensor power spectrum and $n_{\rm T}$ is the tensor spectral index. The transfer function $\mathcal{M}(k,\,k_{\rm eq},\,\tau)$ can be computed under several approximations depending upon the required accuracy. The transfer function for the amplitude of the tensor modes can be numerically computed by solving the evolution of the tensor fluctuations across the matter-radiation equality and the result is [40, 41]

$$\mathcal{M}(k,\,k_{\rm eq},\,\tau) = \frac{9\, j_1^2(k\tau)}{|k\tau|^2}\biggl[1 + c_1\biggl(\frac{k}{k_{\rm eq}}\biggr) + c_2\biggl(\frac{k}{k_{\rm eq}}\biggr)^2\biggr], \tag{5.10}$$

where[15], according to [40, 41], $c_1 = 1.26$ and $c_2 = 2.68$. In Eq. (5.10) $j_1(y) = (\sin y/y^2 - \cos y/y)$ is the spherical Bessel function of first kind which is related to the approximate solution of the evolution equations for the tensor mode functions whenever the solutions are computed deep in the matter-dominated phase (i.e. $a(\tau) \simeq \tau^2$). Instead of working directly with $\mathcal{A}_{\rm T}$ it is often preferred to introduce the quantity customarily called $r_{\rm T}$ denoting the ratio between the tensor and the scalar amplitude at the pivot scale $k_{\rm p}$

$$r_{\rm T} = \frac{\mathcal{A}_{\rm T}}{\mathcal{A}_{\mathcal{R}}} = \frac{\overline{\mathcal{P}}_{\rm T}(k_{\rm p})}{\mathcal{P}_{\mathcal{R}}(k_{\rm p})}. \tag{5.11}$$

[14]Following the standard notations (slightly modified to avoid possible clashes with previously defined variables) $\Omega_{\rm b},\,\Omega_{\rm c},\Omega_{\rm de}$ denote, respectively, the present critical fractions of thebaryons, of the dark matter, of the dark energy; h_0 is the Hubble constant in units of $100\,\mathrm{km/(sec\,Mpc)}$, $n_{\rm s}$ is the scalar spectral index while $\epsilon_{\rm re}$ denotes the optical depth at recombination.

[15]The analysis of [42] gave $c_1 = 1.34$ and $c_2 = 2.50$ which is fully compatible with the results of [40, 41]. In the approach of [42] (see also [43]) the calculation of the amplitude transfer function, in fact, involve a delicate matching on the phases of the tensor mode functions. Conversely, if the transfer function is computed directly for the spectral energy density, the oscillatory contributions are suppressed as the wavelengths get shorter than the Hubble radius (see below).

In principle $n_{\rm T}$ can be taken to be independent of $r_{\rm T}$ and this possibility will also be contemplated in the present discussion. At the same time, if the scalar and the tensor modes are both of inflationary origin, $n_{\rm T}$ is related to $r_{\rm T}$ and to the slow-roll parameter ϵ which measure the rate of decrease of the Hubble parameter during the conventional inflationary stage of expansion:

$$n_{\rm T} = -\frac{r_{\rm T}}{8} = -2\epsilon, \qquad \epsilon = -\frac{\dot{H}}{H^2}; \tag{5.12}$$

the overdot denotes the usual derivative with respect to the cosmic time coordinate; in Eq. (5.12) the spectral index is frequency-independent but there exist situations where more general possibilities can be contemplated such as, for instance

$$n_{\rm T} = -2\epsilon + \frac{\alpha_{\rm T}}{2}\ln{(k/k_{\rm p})}, \qquad \alpha_{\rm T} = \frac{r_{\rm T}}{8}\biggl[(n_{\rm s} - 1) + \frac{r_{\rm T}}{8}\biggr]. \tag{5.13}$$

If $\alpha_{\rm T} = 0$ the tensor spectral index $n_{\rm T}$ does not depend upon the frequency and this is the case which is, somehow, endorsed when introducing gravitational waves in the minimal tensor extension of the ΛCDM. If a tensor component is allowed in the analysis of the WMAP 7yr data alone the relevant cosmological parameters are determined to be

$$(\Omega_{\rm b},\, \Omega_{\rm c}, \Omega_{\rm de},\, h_{0},\, n_{\rm s},\, \epsilon_{\rm re}) \equiv (0.0430,\, 0.200,\, 0.757,\, 0.735,\, 0.982,\, 0.091). \tag{5.14}$$

In the case of Eq. (5.7) the amplitude of the scalar modes is $\mathcal{A}_{\mathcal{R}} = (2.43 \pm 0.11) \times 10^{-9}$ while in the case of Eq. (5.14) the corresponding values of $\mathcal{A}_{\mathcal{R}}$ and of $r_{\rm T}$ are given by

$$\mathcal{A}_{\mathcal{R}} = (2.28 \pm 0.15) \times 10^{-9}, \qquad r_{\rm T} < 0.36, \tag{5.15}$$

to 95 % confidence level. To avoid confusions it is appropriate to spend a word of care on the figures implied by Eqs. (5.14) and (5.15) which have been used in the numeric analysis just for sake of accuracy. The qualitative features of the effects discussed here do not change if, for instance, one would endorse the parameters drawn from the comparison of the minimal tensor extension of the ΛCDM with the WMAP 5yr data release [44, 45], implying, for instance, $\mathcal{A}_{\mathcal{R}} = 2.1^{+2.2}_{-2.3} \times 10^{-9}$, $n_{\rm s} = 0.984$ and $r_{\rm T} < 0.65$ (95 % confidence level). Similar orders of magnitude can be also obtained from even older releases [46, 47].

6 Magnetic field induced by the total vorticity

The total vorticity derived in the previous sections is larger than the vorticity of the ions. Therefore, the total magnetic field derived on the basis of $\omega_{\rm tot}^{i}$ is larger than the one derived on the basis of the ion contribution. Of course this statement holds in an averaged sense since what matters is not the vorticity itself but rather its two-point function which will be explicitly computed in the present section. Using Eq. (5.5) the maximal obtainable magnetic

field will be the one given by Eqs. (3.25)–(3.26) or (3.30) where the total vorticity induced by the geometry is given by Eq. (5.5)

$$B^{i}_{\mathrm{max}}(\vec{x},\tau) = -\frac{\rho_{\mathrm{i}}\sqrt{\gamma}}{e\, N^2 \tilde{n}_{\mathrm{i}}} \omega^{i}_{\mathrm{tot}}(\vec{x},\tau). \tag{6.1}$$

which can also be written, by explicitly keeping track of the number of gradients, as

$$B^{i}_{\mathrm{max}}(\vec{x},\tau) = \biggl\{\mathcal{L}(\tau,w)\, \epsilon^{mij}\partial_{j}\biggl[h^{a\ell}h_{\ell}^{b}\partial_{b}h_{am} + h_{mq}h^{ba}\partial_{b}h^{q}_{a}\biggr] + \mathcal{O}(\epsilon^3)\biggr\}\, a(\tau)\, \biggl[1 + \mathcal{O}(\epsilon^2)\biggr]. \tag{6.2}$$

The prefactor appearing in Eq. (6.1) has been estimated, in Eq. (6.2), by recalling that, to lowest order in the gradient expansion

$$\partial_{\tau}\rho_{\mathrm{i}} = NK\rho_{\mathrm{i}}, \qquad \partial_{\tau}\tilde{n}_{\mathrm{i}} = NK\tilde{n}_{\mathrm{i}}, \tag{6.3}$$

implying that ρ_{i} and $\tilde{n}_{\mathrm{i}}$ scale in the same way with $\sqrt{\gamma}$ since $NK = -\partial_{\tau}\ln\sqrt{\gamma}$. But then, from Eqs. (4.30), (4.31) and (5.2):

$$\frac{\rho_{\mathrm{i}}\sqrt{\gamma}}{N^2\tilde{n}_{\mathrm{i}}} = a(\tau)\biggl[1 + \mathcal{O}(\epsilon^2)\biggr], \tag{6.4}$$

where the first correction is $\mathcal{O}(\epsilon^2)$ and depends on β (see, e.g. Eqs. (4.30) and (4.31)) but it will be immaterial for the present ends. From now on the subscripts will be dropped but it will be always understood that we are referring here to the total vorticity and to the maximally achievable magnetic field. As a consequence of Eq. (6.1) the correlation function of the magnetic field can be related to the correlation function of the vorticity. To estimate the correlation of the vorticity and to obtain an explicit expression the key point is to reduce the six-point function of the tensor modes to the product of two-point functions. For this purpose it is not sufficient to consider the trace of the two-point function introduced in Eq. (5.8) but it is rather necessary to proceed with the full tensorial structure of the correlator whose general parity and rotationally-invariant form will be denoted as

$$G_{ijmn}(r) = \langle h_{ij}(\vec{x},\tau)\, h_{mn}(\vec{y},\tau)\rangle, \tag{6.5}$$

where $G_{ijmn}(r)$ is only function of $r = |\vec{r}|$ where $\vec{r} = \vec{x} - \vec{y}$. Since both $h_{ij}(\vec{x},\tau)$ and $h_{mn}(\vec{y},\tau)$ are transverse and traceless, $G_{ijmn}(r)$ will have to share the same properties. In particular, $G_{ijmn}(r)$ must be symmetric for $i \to j$, $m \to n$, $(i\,j) \to (m\,n)$ and satisfy the following properties

$$\frac{\partial}{\partial r^{i}}G_{ijmn} = 0, \qquad G_{iimn} = G_{ijmm} = 0 \tag{6.6}$$

$$\mathrm{Tr}[G_{ijmn}] = G_{ijij} = \int \frac{dk}{k}\mathcal{P}_{\mathrm{T}}(k)\,\frac{\sin kr}{kr}. \tag{6.7}$$

The properties of Eq. (6.6) and (6.7) are a reflection of the divergenceless and traceless nature of $h_{ij}(\vec{x},\tau)$ while the requirement on the trace follows from the consistency with Eq. (5.8). The general form of G_{ijmn} can therefore be written as

$$\begin{aligned} G_{ijmn}(r) &= \Big(\delta_{im}\delta_{nj} + \delta_{mj}\delta_{ni}\Big) G_1(r) + \delta_{ij}\delta_{mn}\, G_2(r) \\ &+ \Big(\delta_{ij}\, r_m\, r_n + \delta_{mn}\, r_i\, r_j\Big) G_3(r) \\ &+ \Big(\delta_{jn} r_i r_m + \delta_{im} r_j r_n + \delta_{jm} r_i r_n + \delta_{in} r_j r_m\Big) G_4(r) \\ &+ r_i\, r_j\, r_m\, r_n G_5(r), \end{aligned} \tag{6.8}$$

where the various independent functions appearing in Eq. (6.8) are determined in appendix B. The methods used to analyze the real-space correlators are the ones exploited in usual applications of statistical fluid mechanics [48, 49]. To evaluate Eq. (6.2) can then proceed as follows. Using Eq. (5.5), the explicit form of the correlator of the vorticity becomes

$$\begin{aligned} \langle \omega^i(\vec{x},\tau)\, \omega^i(\vec{y},\tau)\rangle &= \mathcal{L}^2(\tau,w) \epsilon^{jmi}\, \epsilon^{j'm'i} \frac{\partial^2}{\partial y^{j'}\, \partial y^{b'}} \frac{\partial^2}{\partial x^j\, \partial x^b} \mathcal{T}_{bmb'm'} \\ \mathcal{T}_{bmb'm'} &= \langle \Big[h_{a\,b}\, h_{a\,q}\, h_{q\,m}\Big]_{\vec{x}} \Big[h_{a'\,b'} h_{a'\,q'} h_{q'\,m'}\Big]_{\vec{y}} \rangle. \end{aligned} \tag{6.9}$$

By defining $\langle \omega^2(r)\rangle = \langle \omega^i(\vec{x},\tau)\, \omega^i(\vec{y},\tau)\rangle$ and by recalling the notations of appendix B, we shall have that[16]

$$\langle \omega^2(r)\rangle = \mathcal{L}^2(\tau,w) \epsilon^{jmi}\, \epsilon^{j'm'i} \frac{\partial^2}{\partial r^{j'}\, \partial r^{b'}} \frac{\partial^2}{\partial r^j\, \partial r^b} \mathcal{T}_{m'b'mb}, \tag{6.10}$$

where the quantity $\mathcal{T}_{m'b'mb}$ is a function of r; the explicit form of $\mathcal{T}_{m'b'mb}$ is given in appendix B in terms of the two-point functions G_{ijmn}. Furthermore, since $T_{m'b'mb}$ can be written, in general terms, as

$$\begin{aligned} \mathcal{T}_{m'b'mb} &= T_1(r)(\delta_{m'b}\delta_{mb'} + \delta_{mm'}\delta_{bb'}) + T_2(r)\delta_{m'b'}\delta_{mb} \\ &+ T_3(r)(\delta_{m'b'} r_m r_b + \delta_{mb} r_{m'} r_{b'}) + T_4(r)(\delta_{bm'} r_m r_{b'} + \delta_{mb'} r_b r_{m'} \\ &+ \delta_{mm'} r_b r_{b'} + \delta_{bb'} r_m r_{m'}) + r_{m'} r_{b'} r_m r_b T_5(r). \end{aligned} \tag{6.11}$$

By using the results of appendix B the explicit values of the five $T_i(r)$ can be expressed in terms of the two-point functions G_{ijmn} (see Eq. (B.22) and (B.23)–(B.27)). There are two physically complementary regimes where the primordial vorticity and hence the magnetic field can be evaluated. Comoving lengths r_{G} defined between 1 and 100 Mpc are smaller than the Hubble radius at equality since[17]

$$r_{\mathrm{eq}} = \frac{2(\sqrt{2}-1)}{H_0} \frac{\sqrt{\Omega_{\mathrm{R0}}}}{\Omega_{\mathrm{M0}}} = 119.397 \Big(\frac{h_0^2 \Omega_{\mathrm{M0}}}{0.134}\Big)^{-1} \Big(\frac{h_0^2 \Omega_{\mathrm{R0}}}{4.15\times 10^{-5}}\Big)^{1/2}\ \mathrm{Mpc}, \tag{6.12}$$

[16] Recall that $\vec{r} = \vec{x} - \vec{y}$, i.e. $r = |\vec{r}| = |\vec{x} - \vec{y}|$.

[17] The quantity r_{T} (denoting, in sec. 5, the tensor to scalar ratio) must not be confused with r_{G} and r_{eq} which denote specific values of the radial coordinate.

where H_0 is the present value of the Hubble rate, Ω_{M0} is the present value of the critical fraction in matter and Ω_{R0} is the present value of the critical fraction in radiation. The pivot length $r_{\mathrm{p}} = 500\,\mathrm{Mpc}$ at which the tensor amplitudes are assigned is such that $r_{\mathrm{G}} < r_{\mathrm{eq}} < r_{\mathrm{p}}$. Therefore, after matter-radiation equality and, in particular, at photon decoupling, the correlation function of the magnetic field can be estimated as

$$\begin{aligned}
\langle B^2(r)\rangle &= 6.348\times 10^{-76}\biggl(\frac{r_{\mathrm{T}}}{0.32}\biggr)^3\biggl(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\biggr)^3\biggl(\frac{z_{\mathrm{dec}}+1}{1089.2}\biggr)^2 \\
&\times \biggl(\frac{h_0^2\Omega_{\mathrm{M0}}}{0.134}\biggr)^6\biggl(\frac{h_0^2\Omega_{\mathrm{R0}}}{4.15\times 10^{-5}}\biggr)^{-6}\,\mathcal{C}(n_{\mathrm{T}},r)\ \mathrm{G}^2, \\
\mathcal{C}(n_{\mathrm{T}},r) &= c(n_{\mathrm{T}})\biggl(\frac{r}{r_{\mathrm{p}}}\biggr)^{8-3n_{\mathrm{T}}} + d(n_{\mathrm{T}}),
\end{aligned} \tag{6.13}$$

in units of $\mathrm{G}^2 \equiv \mathrm{Gauss}^2$ and where the constants $c(n_{\mathrm{T}})$ and $d(n_{\mathrm{T}})$ are given by[18]

$$\begin{aligned}
c(n_{\mathrm{T}}) &= -2(n_{\mathrm{T}}-4)(n_{\mathrm{T}}-3)[2n_{\mathrm{T}}(n_{\mathrm{T}}-6)+19]\cos^3\biggl(\frac{n_{\mathrm{T}}\pi}{2}\biggr)\Gamma^3(n_{\mathrm{T}}-5), \\
d(n_{\mathrm{T}}) &= -\frac{36+14n_{\mathrm{T}}(n_{\mathrm{T}}-4)}{45n_{\mathrm{T}}(n_{\mathrm{T}}^2-6n_{\mathrm{T}}+8)^2}.
\end{aligned} \tag{6.14}$$

The typical values of n_{T} are negative and $\mathcal{O}(10^{-2})$. Indeed, assume, consistently with Eq. (5.15), that $r_{\mathrm{T}} \sim 0.32$. Then, according to Eq. (5.12), $n_{\mathrm{T}} \sim -0.04$ and $\epsilon \sim 0.02$. Concerning the results of Eqs. (6.13) and (6.14) few comments are in order:

- the prefactor $\mathcal{L}(\tau,w)$ is estimated in the hypothesis $w=0$, $a_1 = a_{\mathrm{eq}}$ and $\mathcal{H}_1 = \mathcal{H}_{\mathrm{eq}}$ since we ought to estimate the field prior to photon decoupling;
- recalling that $\mathcal{H} = aH$ the value of the Hubble rate at the equality time can be estimated as:
$$H_{\mathrm{eq}} = \sqrt{2\ \Omega_{\mathrm{M0}}}\,H_0\biggl(\frac{a_0}{a_{\mathrm{eq}}}\biggr)^{3/2} \equiv 1.65\times 10^{-56}\biggl(\frac{h_0^2\Omega_{\mathrm{M0}}}{0.134}\biggr)^2 M_{\mathrm{P}}; \tag{6.15}$$
- the result of Eq. (6.13) holds for comoving scales $r < r_{\mathrm{P}} = 500$ Mpc (which are the ones relevant for the gravitational collapse of the protogalaxy) and it is not sensitive to the variation of r provided n_{T} is nearly scale-invariant;
- if $r_{\mathrm{T}} \simeq 0.32$, then $n_{\mathrm{T}} = -0.04$; from Eq. (6.14), $\mathcal{C}(r_{\mathrm{G}}, -0.04) \simeq 0.07$ while for $r = 100\,r_{\mathrm{G}}$ we have that $\mathcal{C}(100\,r_{\mathrm{G}}, -0.04) \simeq 0.01$.

By thus approximating $\mathcal{C}(n_{\mathrm{T}},r) \simeq \mathcal{O}(1)$ in the range $r = 1$–100 Mpc and for $0.2 < r_{\mathrm{T}} < 0.3$ we get the following value for $B_{\mathrm{max}} = \sqrt{\langle B^2(r)\rangle}$

$$\begin{aligned}
B_{\mathrm{max}} &= 2.519\times 10^{-38}\biggl(\frac{r_{\mathrm{T}}}{0.32}\biggr)^{3/2}\biggl(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\biggr)^{3/2}\biggl(\frac{z_{\mathrm{dec}}+1}{1089.2}\biggr) \\
&\times \biggl(\frac{h_0^2\Omega_{\mathrm{M0}}}{0.134}\biggr)^3\biggl(\frac{h_0^2\Omega_{\mathrm{R0}}}{4.15\times 10^{-5}}\biggr)^{-3}\ \mathrm{G}.
\end{aligned} \tag{6.16}$$

[18]Recall that because of the relation (5.12) $n_{\mathrm{T}} < 0$ and $r_{\mathrm{T}} > 0$.

The result of Eq. (6.16) does not seem to be even remotely relevant for galactic magnetogenesis or for cluster magnetogenesis. In spite of the intricacy and of the ramification of the galactic dynamo hypothesis, it is useful to compare Eq. (6.16) with the minimal requirements stemming from what we would call optimal or ideal dynamo, namely a process where the kinetic energy of the protrogalaxy is converted into magnetic energy with maximal efficiency. Let us denote with N_{rot} the number of (effective) rotations performed by the galaxy since gravitational collapse and with ρ_{a} and ρ_{b} the matter density after and before gravitational collapse .

The typical rotation period of a spiral galaxy is of the order of 3×10^8 yrs which should be compared with 10^{10} yrs, i.e. the approximate age of the galaxy. The maximal number of rotations performed by the galaxy since its origin is then of the order of $N_{\mathrm{rot}} \sim 30$. Under the hypothesis that the kinetic energy of the plasma is transferred to the magnetic energy with maximal efficiency, the protogalactic field will be amplified by one efold during each rotation. The effective number of efolds is however always smaller than 30 for various reasons. Typically it can happen that the dynamo quenches prematurely because some of the higher wavenumbers of the magnetic field become critical (i.e. comparable with the kinetic energy of the plasma) before the smaller ones. Other sources of quenching have been recently discussed in the literature (see, for an introduction to this topic, section 4.2 of [50] and references therein). There is also another source of amplification of the primordial magnetic field and it has to do with compressional amplification. At the time of the gravitational collapse of the protogalaxy the conductivity of the plasma was sufficiently high to justify the neglect of nonlinear corrections in the equations expressing the conservation of the magnetic flux and of the magnetic helicity. The conservation of the magnetic flux implies that, during the gravitational collapse, the magnetic field should undergo compressional amplification, i.e. the same kind of mechanism which is believed to be the source of the large magnetic fields of the pulsars. Taking into account the two previous observations the estimate of Eq. (6.16) must be compared with the bound

$$B_{\mathrm{bound}} \simeq 3 \times 10^3 \, e^{-N_{\mathrm{rot}}} \left(\frac{\rho_{\mathrm{b}}}{\rho_{\mathrm{a}}}\right)^{2/3} \mathrm{nG} \tag{6.17}$$

in nG units. Even assuming $N_{\mathrm{rot}} = 30$, $\rho_{\mathrm{a}} \simeq 10^{-24}\,\mathrm{g/cm^3}$, and $\rho_{\mathrm{b}} \simeq 10^{-29}\,\mathrm{g/cm^3}$ the minimal value of B_{bound} is $\mathcal{O}(10^{-25})$G. Clearly, by comparing Eq. (6.16) with Eq. (6.17), $B_{\max} \ll B_{\mathrm{bound}}$.

Going then to cluster magnetogenesis, the typical scale of the gravitational collapse of a cluster is larger (roughly by one order of magnitude) than the scale of gravitational collapse of the protogalaxy. The mean mass density within the Abell radius ($\simeq 1.5 h_0^{-1}$ Mpc) is roughly 10^3 times larger than the critical density since clusters are formed from peaks in the density field. Moreover, clusters rotate much less than galaxies even if it is somehow hard to disentangle, observationally, the global (coherent) rotation of the cluster from the rotation curves of the constituent galaxies. By assuming, for instance, $N_{\mathrm{rot}} = 5$, a density gradient of 10^3 and 500 nG as final field, Eq. (6.17) demands and initial seed of the order 0.15 nG.

Another application of the results obtained in the previous sections can be the estimate of the magnetic field induced by the total vorticity for scales which are larger than the Hubble radius prior to matter radiation equality. To conduct this estimate the explicit form of the correlators will change. First of all in the pre-factor $\mathcal{L}(\tau, w)$ we shall choose $w = 1/3$ and $a_1 = a_{\mathrm{r}}$ and $\mathcal{H}_1 = \mathcal{H}_{\mathrm{r}}$ with $H_{\mathrm{r}} \simeq 10^{-5}\,M_{\mathrm{P}}$. Thus for typical length-scales larger than the Hubble radius at equality and for typical times of the order of the equality time the analog of Eq. (6.13) can be written as

$$
\begin{aligned}
\langle B^2(r)\rangle &= 2.915\times 10^{-79}\biggl(\frac{r_{\mathrm{T}}}{0.32}\biggr)^{3}\biggl(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\biggr)^{3}\biggl(\frac{z_{\mathrm{dec}}+1}{1089.2}\biggr)^{2}\\
&\times \biggl(\frac{h_0^2\Omega_{\mathrm{M0}}}{0.134}\biggr)^{-4}\mathcal{C}(n_{\mathrm{T}}, r)\ \mathrm{G}^2,\\
\mathcal{C}(n_{\mathrm{T}}, r) &= \tilde{c}(n_{\mathrm{T}})\biggl(\frac{r}{r_{\mathrm{p}}}\biggr)^{-4-3n_{\mathrm{T}}} + \tilde{d}(n_{\mathrm{T}}),
\end{aligned}
\tag{6.18}
$$

where the numerical constants $\tilde{c}(n_{\mathrm{T}})$ and $\tilde{d}(n_{\mathrm{T}})$ are given by

$$
\begin{aligned}
\tilde{c}(n_{\mathrm{T}}) &= -2n_{\mathrm{T}}(n_{\mathrm{T}}+1)[2n_{\mathrm{T}}(n_{\mathrm{T}}+2)+3]\cos^3\biggl(\frac{n_{\mathrm{T}}\pi}{2}\biggr)\Gamma^3(n_{\mathrm{T}}-1),\\
\tilde{d}(n_{\mathrm{T}}) &= -\frac{2(7n_{\mathrm{T}}^2+28n_{\mathrm{T}}+18)}{45n_{\mathrm{T}}^2(n_{\mathrm{T}}+2)^2(n_{\mathrm{T}}+4)}.
\end{aligned}
\tag{6.19}
$$

Equation (6.18) holds under the assumption $r < r_{\mathrm{p}}$ which means, in practice, that it applies only for a narrow range of scales $120\,\mathrm{Mpc} < r < 500\,\mathrm{Mpc}$. If $r \simeq 250\,\mathrm{Mpc}$ then $r/r_{\mathrm{p}} = 0.5$ and $\mathcal{C}(n_{\mathrm{T}}, r) \simeq \mathcal{O}(163)$ and

$$
B_{\mathrm{max}} = 4.3\times 10^{-39}\biggl(\frac{r_{\mathrm{T}}}{0.32}\biggr)^{3/2}\biggl(\frac{\mathcal{A}_{\mathcal{R}}}{2.43\times 10^{-9}}\biggr)^{3/2}\biggl(\frac{h_0^2\Omega_{\mathrm{M0}}}{0.134}\biggr)^{-2}\ \mathrm{G}. \tag{6.20}
$$

7 Concluding remarks

The idea explored in this paper has been to compute the vorticity by employing a recently devised framework for the treatment of fully inhomogeneous plasmas which are also gravitating. The latter description brings a new perspective to the study of the evolution of the vorticity exchange in the electron-ion-photon system without postulating the customary separation between a (preferably conformally flat) background geometry and its relativistic fluctuations. A set of general conservation laws has been derived on the basis of the fully inhomogeneous equations in different temperature regimes depending on the hierarchies between the exchange rate of the vorticity between electrons, ions and photons. After expanding the Einstein equations as well as the vorticity equations to a given order in the spatial gradients, the total vorticity has then been estimated to lowest order in the gradient expansion.

The maximal comoving magnetic field induced in the ΛCDM paradigm depends upon the tensor to scalar ratio and it is, at most, of the order of 10^{-37} G over the typical comoving scales ranging between 1 and 10 Mpc. The obtained results are irrelevant for seeding a reasonable galactic dynamo action and they demonstrate how the proposed fully inhomogeneous treatment can be used for a systematic scrutiny of pre-decoupling plasmas beyond the conventional perturbative expansions. The estimate of the primordial vorticity induced in the ΛCDM scenario can also turn out to be relevant in related contexts such as the ones contemplated by non conventional paradigms of galaxy formation.

A Gradient expansion and pre-decoupling physics

In this appendix we are going to recap the essentials of the fully inhomogeneous description of pre-decoupling plasmas already introduced in Eqs. (3.1)–(3.5). We will follow here the formalism developed in Ref. [14] and describe the fully inhomogeneous geometry in terms of the ADM decomposition [15, 16]:

$$g_{00} = N^2 - N_k N^k, \qquad g_{ij} = -\gamma_{ij}, \qquad g_{0i} = -N_i,$$
$$g^{00} = \frac{1}{N^2}, \qquad g^{ij} = \frac{N^i\,N^j}{N^2} - \gamma^{ij}, \qquad g^{0i} = -\frac{N^i}{N^2}. \tag{A.1}$$

In the ADM variables the extrinsic curvature K_{ij} and the spatial components of the Ricci tensor r_{ij} become:

$$K_{ij} = \frac{1}{2N}\Big[-\partial_\tau \gamma_{ij} +^{(3)} \nabla_i N_j +^{(3)} \nabla_j N_i\Big], \tag{A.2}$$
$$r_{ij} = \partial_m\, {}^{(3)}\Gamma^m_{ij} - \partial_j^{(3)}\Gamma^m_{im} +^{(3)} \Gamma^m_{ij}\, {}^{(3)}\Gamma^n_{mn} -^{(3)} \Gamma^m_{jn}\, {}^{(3)}\Gamma^n_{im}. \tag{A.3}$$

Defining as $T_{\mu\nu}$ as the total energy-momentum tensor of the fluid sources, the contracted form of the Einstein equations reads

$$R_\mu^\nu = \ell_{\mathrm{P}}^2\Big[\Big(T_\mu^\nu - \frac{T}{2}\delta_\mu^\nu\Big)\Big], \qquad T = g^{\mu\nu} T_{\mu\nu} = T_\mu^\mu. \tag{A.4}$$

As in the bulk of the paper we are now going to focus on the situation where the shift vectors vanish and the lapse function is homogeneous but time dependent (i.e. $N(\vec{x},\tau) = N(\tau)$). The $(0\,0)$, $(i\,j)$ and $(0\,i)$ components of Eq. (A.4) become then:

$$\partial_\tau K - N \mathrm{Tr} K^2 + \nabla^2 N = N \ell_{\mathrm{P}}^2 \Big\{\frac{3p+\rho}{2} + (p+\rho)\,u^2\Big\}, \tag{A.5}$$
$$\nabla_i K - \nabla_k K_i^k = N \ell_{\mathrm{P}}^2 u^0\, u_i (p+\rho), \tag{A.6}$$
$$\partial_\tau K_i^j - N K K_i^j - N r_i^j + \nabla_i \nabla^j N = \ell_{\mathrm{P}}^2 N \Big[\frac{p-\rho}{2}\delta_i^j - (p+\rho) u_i u^j\Big], \tag{A.7}$$

where $u^2 = u^i\, u^j \gamma_{ij}$. The electron and ion velocities appearing in Eqs. (3.1) and (3.2) reduce, in the conformally flat case (i.e. $N(\tau) \to a(\tau)$ and $\gamma_{ij}(\vec{x},\tau) \to a^2(\tau)\delta_{ij}$) to the velocity fields appearing in Eqs. (2.3), (2.13) and (2.15). In the fully inhomogeneous case, the evolution equations for the velocities of the electrons, ions and photons can be written, respectively, as

$$\partial_\tau v_{\mathrm{e}}^k + N \partial^k N - \mathcal{G}_j^k v_{\mathrm{e}}^j = -\frac{e \tilde{n}_{\mathrm{e}} N^2}{\rho_{\mathrm{e}} \sqrt{\gamma}}\Big[E^k + (\vec{v}_{\mathrm{e}} \times \vec{B})^k\Big]$$
$$+ N \Gamma_{\mathrm{ei}} (v_{\mathrm{i}}^k - v_{\mathrm{e}}^k) + \frac{4}{3}\frac{\rho_\gamma}{\rho_{\mathrm{e}}} N \Gamma_{\mathrm{e}\gamma} (v_\gamma^k - v_{\mathrm{e}}^k), \tag{A.8}$$
$$\partial_\tau v_{\mathrm{i}}^k + N \partial^k N - \mathcal{G}_j^k v_{\mathrm{i}}^j = \frac{e \tilde{n}_{\mathrm{i}} N^2}{\rho_{\mathrm{i}} \sqrt{\gamma}}\Big[E^k + (\vec{v}_{\mathrm{i}} \times \vec{B})^k\Big]$$

$$
\begin{aligned}
& + \quad N\,\frac{\rho_{\mathrm{e}}}{\rho_{\mathrm{i}}}\,\Gamma_{\mathrm{ie}}(v_{\mathrm{e}}^{k} - v_{\mathrm{i}}^{k}) + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{i}}} N \Gamma_{\mathrm{i}\gamma}(v_{\gamma}^{k} - v_{\mathrm{i}}^{k}), \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\tau} v_{\gamma}^{k} + N \partial^{k} N - \biggl[\mathcal{G}_{j}^{k} - \frac{NK}{3}\delta_{j}^{k}\biggr] v_{\gamma}^{j} &= -\frac{N^2}{4\rho_{\gamma}}\partial_{m}\Bigl(\rho_{\gamma}\gamma^{mk}\Bigr) \\
&+ \quad N\Gamma_{\gamma\mathrm{e}}(v_{\mathrm{e}}^{k} - v_{\gamma}^{k}) + N\Gamma_{\gamma\mathrm{i}}(v_{\mathrm{i}}^{k} - v_{\gamma}^{k}), \qquad \text{(A.10)}
\end{aligned}
$$

where

$$
\mathcal{G}_{j}^{k} = \biggl[\frac{\partial_{\tau} N}{N}\delta_{j}^{k} + 2 N K_{j}^{k}\biggr]. \qquad \text{(A.11)}
$$

As in the conformally flat case the evolution equations of the electrons and of the ions can be combined by defining the center of mass velocity of the electron-ion system $v_{\mathrm{b}}^{k} = (m_{\mathrm{e}} v_{\mathrm{e}}^{k} + m_{\mathrm{i}} v_{\mathrm{i}}^{k})/(m_{\mathrm{e}} + m_{\mathrm{i}})$ so that the the effective evolution equations for the baryon-lepton-photon fluid become

$$
\begin{aligned}
\partial_{\tau}\rho_{\gamma} &= \frac{4}{3} K N \rho_{\gamma} - \frac{4}{3} N \partial_{k}\Bigl(\frac{\rho_{\gamma}}{N}\,v_{\gamma}^{k}\Bigr), && \text{(A.12)}\\
\partial_{\tau} v_{\mathrm{b}}^{k} &= \mathcal{G}_{j}^{k} v_{\mathrm{b}}^{j} - N\partial^{k} N + \frac{(\vec{J}\times\vec{B})^{k} N^2}{\gamma\,\rho_{\mathrm{b}}(1 + m_{\mathrm{e}}/m_{\mathrm{i}})} + \frac{4}{3}\frac{\rho_{\gamma}}{\rho_{\mathrm{b}}} N\Gamma_{\gamma\mathrm{e}}(v_{\gamma}^{k} - v_{\mathrm{b}}^{k}), && \text{(A.13)}\\
\partial_{\tau} v_{\gamma}^{k} &= \biggl[\mathcal{G}_{j}^{k} - \frac{NK}{3}\delta_{j}^{k}\biggr] v_{\gamma}^{j} - \frac{N^2}{4\rho_{\gamma}}\partial_{m}\Bigl(\rho_{\gamma}\gamma^{mk}\Bigr) - N\partial^{k} N + N\Gamma_{\gamma\mathrm{e}}(v_{\mathrm{b}}^{k} - v_{\gamma}^{k}), && \text{(A.14)}
\end{aligned}
$$

where v_{γ}^{k} and ρ_{γ} denote, respectively, the photon velocity and the photon energy density.

B Some relevant correlators

The correlator appearing in Eq. (6.9), i.e.

$$
\mathcal{T}_{bmb'm'} = \langle \Bigl[h_{ab}\,h_{aq}\,h_{qm}\Bigr]_{\vec{x}}\,\Bigl[h_{a'\,b'} h_{a'\,q'} h_{q'm'}\Bigr]_{\vec{y}}\rangle, \qquad \text{(B.1)}
$$

must be computed in terms of the corresponding two-point functions in real space (see Eq. (6.5)). The general form of the two-point function in real space has been already mentioned in Eq. (6.8) and the functions $G_i(r)$ (with $i = 1, \ldots 5$) are given by:

$$
\begin{aligned}
G_1(r) &= F_1(r) + \frac{2}{r}\frac{\partial F_2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_3}{\partial r}\Bigr), && \text{(B.2)}\\
G_2(r) &= \frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_3}{\partial r}\Bigr) - F_1(r) - \frac{2}{r}\frac{\partial F_2}{\partial r}, && \text{(B.3)}\\
G_3(r) &= \frac{1}{r}\frac{\partial}{\partial r}\Bigl[\frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_3}{\partial r}\Bigr)\Bigr] - \frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_2}{\partial r}\Bigr), && \text{(B.4)}\\
G_4(r) &= \frac{1}{r}\frac{\partial}{\partial r}\Bigl[\frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_3}{\partial r}\Bigr)\Bigr] + \frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_2}{\partial r}\Bigr), && \text{(B.5)}\\
G_5(r) &= \frac{1}{r}\frac{\partial}{\partial r}\Bigl\{\frac{1}{r}\frac{\partial}{\partial r}\Bigl[\frac{1}{r}\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial F_3}{\partial r}\Bigr)\Bigr]\Bigr\}, && \text{(B.6)}
\end{aligned}
$$

where $F_1(r)$, $F_2(r)$ and $F_3(r)$ are fully determined once the power spectrum is known and are defined as:

$$
\begin{aligned}
F_1(r) &= \frac{1}{4}\int \frac{dk}{k}\,\mathcal{P}_{\rm T}(k,\tau)\frac{\sin kr}{kr}, \qquad F_2(r) = \frac{1}{4}\int \frac{dk}{k^3}\,\mathcal{P}_{\rm T}(k,\tau)\frac{\sin kr}{kr},\\
F_3(r) &= \frac{1}{4}\int \frac{dk}{k^5}\,\mathcal{P}_{\rm T}(k,\tau)\frac{\sin kr}{kr}.
\end{aligned}
\tag{B.7}
$$

Using Eq. (B.7) inside Eqs. (B.2)–(B.6) we have that

$$
G_1(r) = \frac{1}{4}\int \frac{dk}{k}\,\mathcal{P}_{\rm T}(k,\tau)\Big[\Big(1-\frac{1}{k^2r^2}\Big)j_0(kr) + \Big(\frac{3}{k^2r^2}-2\Big)\frac{j_1(kr)}{kr}\Big], \tag{B.8}
$$

$$
G_2(r) = \frac{1}{4}\int \frac{dk}{k}\,\mathcal{P}_{\rm T}(k,\tau)\Big[\Big(2+\frac{3}{k^2r^2}\Big)\frac{j_1(kr)}{kr} - \Big(1+\frac{1}{k^2r^2}\Big)j_0(kr)\Big], \tag{B.9}
$$

$$
G_3(r) = \frac{1}{4}\int k\,dk\mathcal{P}_{\rm T}(k,\tau)\Big[\frac{j_0(kr)}{k^2r^2}\Big(1+\frac{5}{k^2r^2}\Big) - \frac{j_1(kr)}{k^3r^3}\Big(2+\frac{15}{k^2r^2}\Big)\Big], \tag{B.10}
$$

$$
G_4(r) = \frac{1}{4}\int k\,dk\mathcal{P}_{\rm T}(k,\tau)\Big[\frac{j_0(kr)}{k^2r^2}\Big(-1+\frac{5}{k^2r^2}\Big) + \frac{j_1(kr)}{k^3r^3}\Big(4-\frac{15}{k^2r^2}\Big)\Big], \tag{B.11}
$$

$$
G_5(r) = \frac{1}{4}\int k^3 dk\mathcal{P}_{\rm T}(k,\tau)\Big[\frac{j_0(kr)}{k^4r^4}\Big(1-\frac{35}{k^2r^2}\Big) - \frac{5j_1(kr)}{k^5r^5}\Big(2-\frac{21}{k^2r^2}\Big)\Big], \tag{B.12}
$$

where $j_0(kr)$ and $j_1(kr)$ are spherical Bessel functions of zeroth and first-order [51, 52]

$$
j_0(kr) = \frac{\sin kr}{kr}, \qquad j_1(kr) = \frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}. \tag{B.13}
$$

It is useful to compare the two different asymptotic limits of the various $G_i(r)$, i.e. for $kr < 1$ and for $kr > 1$. In the limit $kr > 1$ we have that:

$$
G_1(r) \to \frac{1}{4}\int \frac{dk}{k}\,\mathcal{P}_{\rm T}(k,\tau)j_0(kr), \qquad G_2(r) \to -G_1(r), \tag{B.14}
$$

$$
G_3(r) \to \frac{1}{4}\int k\,dk\,\mathcal{P}_{\rm T}(k,\tau)\frac{j_0(kr)}{k^2r^2}, \qquad G_4(r) \to -G_3(r), \tag{B.15}
$$

$$
G_5(r) \to \frac{1}{4}\int k^3 dk\,\mathcal{P}_{\rm T}(k,\tau)\frac{j_0(kr)}{k^4r^4}. \tag{B.16}
$$

Conversely, in the limit $kr < 1$, Eqs. (B.8)–(B.12) imply:

$$
G_1(r) \to \frac{1}{10}\int \frac{dk}{k}\mathcal{P}_{\rm T}(k,\tau)\Big[1-\frac{11}{42}k^2r^2\Big], \tag{B.17}
$$

$$
G_2(r) \to -\frac{1}{15}\int \frac{dk}{k}\mathcal{P}_{\rm T}(k,\tau)\Big[1-\frac{5k^2r^2}{14}\Big], \tag{B.18}
$$

$$
G_3(r) \to -\frac{2}{105}\int kdk\mathcal{P}_{\rm T}(k,\tau)\Big[1-\frac{5}{72}k^2r^2\Big], \tag{B.19}
$$

$$
G_4(r) \to \frac{1}{70}\int kdk\mathcal{P}_{\rm T}(k,\tau)\Big[1-\frac{2\,k^2r^2}{27}\Big], \tag{B.20}
$$

$$
G_5(r) \to \frac{1}{3780}\int k^3\,dk\mathcal{P}_{\rm T}(k,\tau)\Big[1-\frac{k^2r^2}{22}\Big]. \tag{B.21}
$$

The explicit form of the two-point function G_{ijmn} implies that the six-point function appearing in the correlator of the vorticity can be expressed as

$$\mathcal{T}_{bmb'm'} = \sum_{\nu=1}^{5} \mathcal{T}^{(\nu)}_{b\,m\,b'\,m'}, \tag{B.22}$$

where the 5 distinct contributions correspond to

$$\mathcal{T}^{(1)}_{b\,m\,b'\,m'} = \overline{G}_{abaq}\biggl[G_{qma'b'}\overline{G}_{a'q'q'm'} + G_{qma'q'}\overline{G}_{b'a'q'm'} + G_{qmq'm'}\overline{G}_{a'q'a'b'}\biggr], \tag{B.23}$$

$$\mathcal{T}^{(2)}_{b\,m\,b'\,m'} = \overline{G}_{abqm}\biggl[G_{aqa'b'}\overline{G}_{a'q'q'm'} + G_{aqa'q'}\overline{G}_{a'b'q'm'} + G_{aqq'm'}\overline{G}_{a'b'a'q'}\biggr], \tag{B.24}$$

$$\begin{aligned}\mathcal{T}^{(3)}_{b\,m\,b'\,m'} &= G_{aba'b'}\overline{G}_{aqqm}\overline{G}_{a'q'q'm'} \\ &+ G_{aba'b'}\biggl(G_{aqa'q'}G_{qmq'm'} + G_{aqq'm'}G_{qma'q'}\biggr),\end{aligned} \tag{B.25}$$

$$\begin{aligned}\mathcal{T}^{(4)}_{b\,m\,b'\,m'} &= G_{abq'a'}\overline{G}_{aqqm}\overline{G}_{a'b'q'm'} \\ &+ G_{abq'a'}\biggl(G_{aqa'b'}G_{qmq'm'} + G_{aqq'm'}G_{qma'b'}\biggr),\end{aligned} \tag{B.26}$$

$$\begin{aligned}\mathcal{T}^{(5)}_{b\,m\,b'\,m'} &= G_{abq'm'}\overline{G}_{aqqm}\overline{G}_{a'b'a'q'} \\ &+ G_{abq'm'}\biggl(G_{aqa'b'}G_{qma'q'} + G_{aqa'q'}G_{qma'b'}\biggr).\end{aligned} \tag{B.27}$$

The overline signifies that the corresponding correlator is evaluated in the limit $r \to 0$. According to Eqs. (B.17)–(B.21) this limit is non-singular. Notice finally that in terms of $\mathcal{T}_{bmb'm'}$ the correlation function of the magnetic field can also be written, with shorthand notation, as

$$\begin{aligned}\langle B^2(r)\rangle &= \mathcal{J}(\tau, w)\epsilon^{jmi}\,\epsilon^{j'm'i}\frac{\partial^2}{\partial y^{j'}\,\partial y^{b'}}\frac{\partial^2}{\partial x^{j}\,\partial x^{b}}\mathcal{T}_{bmb'm'} \\ \mathcal{J}(\tau, w) &= \frac{m_{\mathrm{i}}^2}{\alpha_{\mathrm{em}}}\frac{\mathcal{H}^2\,a_1^2}{9\mathcal{H}_1^4(w+1)^2}\biggl(\frac{a}{a_1}\biggr)^{6w+4}.\end{aligned} \tag{B.28}$$

The real space approach is more effective and convenient for an explicit estimate of the vorticity and the idea is therefore to express the correlation functions in real space, take the appropriate derivatives and then expand the result in the desired limit. Denoting with $R = r/r_{\mathrm{p}}$ and with $x = k/k_{\mathrm{p}}$, the integrals over k appearing in Eqs. (B.8)–(B.12) can be computed explicitly by changing variable and by using the following pair of relations [53]:

$$\begin{aligned}\int_1^\infty x^{n-m}\sin xR\,dx &= \frac{R}{m-n-2}\;{}_1F_2\biggl[a_1; b_1, b_2; -\frac{R^2}{4}\biggr] \\ &+ R^{m-n-1}\cos\biggl[\frac{\pi(m-n)}{2}\biggr]\Gamma[1-m+n],\end{aligned} \tag{B.29}$$

$$\begin{aligned}\int_1^\infty x^{n-m}\cos xR\,dx &= \frac{1}{m-n-2}\;{}_1F_2\biggl[\tilde{a}_1; \tilde{b}_1, \tilde{b}_2; -\frac{R^2}{4}\biggr] \\ &+ R^{m-n-1}\sin\biggl[\frac{\pi(m-n)}{2}\biggr]\Gamma[1-m+n],\end{aligned} \tag{B.30}$$

where $n < m$ (i.e. $(n-m)$ is negative). In Eqs. (B.29) and (B.30) ${}_pF_q[a_1....a_p; b_1, ...b_q; z]$ denotes the generalized hypergeometric function of argument z; in the case of Eqs. (B.29) and (B.30) , $p = 1$, $q = 2$ and

$$a_1 = 1 + \frac{n-m}{2}, \qquad b_1 = \frac{3}{2}, \qquad b_2 = 2 + \frac{n-m}{2}, \tag{B.31}$$

$$\tilde{a}_1 = a_1 - \frac{1}{2}, \qquad \tilde{b}_1 = b_1 - 1, \qquad \tilde{b}_2 = b_2 - \frac{1}{2}. \tag{B.32}$$

The integrals are taken from 1 to infinity since the integral over k starts from k_{p} implying that the lower limit of integration in x is 1. Equations (B.29) and (B.30) can be used to derive the real space form of the correlator of Eq. (6.5). Using Eqs. (B.29) and (B.30), the explicit form of Eqs. (B.8)–(B.12) can be derived and inserted into Eq. (6.8) whose explicit form determines the real-space expression of the two-point functions of the vorticities through Eqs. (B.23)–(B.27). After taking the appropriate derivatives the obtained result can be expanded in the wanted limits (e. g. $R \gg 1$ or $R \ll 1$). The explicit real-space expressions of Eqs. (B.8)–(B.12) are typically rather lengthy but they are conceptually straightforward. This is why they will be omitted and only an example will be given. Even if the scales interesting for section 6 will be the ones close to the galactic scale, consider, for instance the expression of $G_1(R)$ in the opposite limit, i.e. comoving scales much larger than r_{eq}. In this case the expressions simplify since $\mathcal{M}(k, k_{\mathrm{eq}}, \tau) \to 1$. Therefore, using Eqs. (B.29) and (B.30), Eq. (B.8) becomes:

$$\begin{aligned}
G_1(R) &= \frac{\mathcal{A}_{\mathrm{T}}}{4}\biggl\{\frac{1}{n_{\mathrm{T}}}\,{}_1F_2\biggl[\frac{n_{\mathrm{T}}}{2}; \frac{3}{2}, \frac{n_{\mathrm{T}}}{2}; -\frac{R^2}{4}\biggr] \\
&+ \frac{3}{4(n_{\mathrm{T}}-4)R^4}\biggl[{}_1F_2\biggl[-2 + \frac{n_{\mathrm{T}}}{2}; \frac{1}{2}, -1 + \frac{n_{\mathrm{T}}}{2}; -\frac{R^2}{4}\biggr] \\
&- {}_1F_2\biggl[-2 + \frac{n_{\mathrm{T}}}{2}; \frac{3}{2}, -1 + \frac{n_{\mathrm{T}}}{2}; -\frac{R^2}{4}\biggr]\biggr] \\
&+ \frac{1}{4(n_{\mathrm{T}}-2)R^2}\biggl[3\,{}_1F_2\biggl[-1 + \frac{n_{\mathrm{T}}}{2}; \frac{3}{2}, \frac{n_{\mathrm{T}}}{2}; -\frac{R^2}{4}\biggr] \\
&- 2\,{}_1F_2\biggl[-1 + \frac{n_{\mathrm{T}}}{2}; \frac{1}{2}, \frac{n_{\mathrm{T}}}{2}; -\frac{R^2}{4}\biggr]\biggr] \\
&- \frac{1}{4R^{n_{\mathrm{T}}}}\cos\left(\frac{\pi n_{\mathrm{T}}}{2}\right)\biggl[\Gamma[n_{\mathrm{T}}-4] + \Gamma[n_{\mathrm{T}}-2] + 3\Gamma[n_{\mathrm{T}}-5] \\
&+ 3\Gamma[n_{\mathrm{T}}-3] + \Gamma[n_{\mathrm{T}}-1]\biggr]\biggr\}.
\end{aligned} \tag{B.33}$$

To conclude this appendix let us show that the expression of G_{ijmn} given in Eq. (6.5) coincides with the result directly obtainable in the particular case where the tensor modes of the geometry are quantized in terms of gravitons. When $h_{ij}(\vec{x}, \tau)$ is a field operator its expression can be written as [40, 41]

$$\hat{h}_{ij}(\vec{x}, \tau) = \frac{\sqrt{2}\ell_{\mathrm{P}}}{(2\pi)^{3/2}a(\tau)} \sum_{\lambda} \int d^3k\, \epsilon_{ij}^{(\lambda)}(\hat{k}) \left[\hat{a}_{\vec{k},\lambda}\, f_{k,\lambda}(\tau)e^{-i\vec{k}\cdot\vec{x}} + \hat{a}^{\dagger}_{\vec{k},\lambda}\, f^{*}_{k,\lambda}(\tau)e^{i\vec{k}\cdot\vec{x}}\right]. \tag{B.34}$$

which also implies, using the properties of the creation and annihilation operators,

$$G_{ijmn}(r) = \langle \hat{h}_{ij}(\vec{x},\tau)\, \hat{h}_{mn}(\vec{y},\tau)\rangle = \int \frac{dk}{k} \mathcal{P}_{\mathrm{T}}(k,\tau)\, \mathcal{Q}_{ijmn}(\hat{k})\, j_0(kr), \tag{B.35}$$

where

$$\mathcal{P}_{\mathrm{T}}(k,\tau) = 4\ell_{\mathrm{P}}^2 \frac{k^3}{\pi^2 a^2(\tau)} |f_k(\tau)|^2, \tag{B.36}$$

$$\mathcal{Q}_{ijmn} = \frac{1}{4}\sum_{\lambda} \epsilon_{ij}^{(\lambda)}\, \epsilon_{mn}^{(\lambda)} = \frac{1}{4}\Big[P_{mi}(\hat{k})P_{nj}(\hat{k}) + P_{mj}(\hat{k})P_{ni}(\hat{k}) - P_{ij}(\hat{k})P_{mn}(\hat{k})\Big]; \tag{B.37}$$

$P_{ij}(\hat{k}) = (\delta_{ij} - \hat{k}_i\hat{k}_j)$ with $\hat{k}^i = k^i/|\vec{k}|$. In Eq. (B.35) it has been used that

$$\langle 0|\, \hat{a}_{\vec{p},\mu}\, \hat{a}^{\dagger}_{\vec{p},\lambda}|0\rangle = \delta^{(3)}(\vec{k} - \vec{p})\delta_{\lambda\mu}. \tag{B.38}$$

Furthermore, to derive Eq. (B.37), the explicit form of the two tensor polarizations can be written, in explicit terms, as

$$\epsilon_{ij}^{(\oplus)}(\hat{k}) = (\hat{a}_i\hat{a}_j - \hat{b}_i\hat{b}_j), \qquad \epsilon_{ij}^{(\otimes)}(\hat{k}) = (\hat{a}_i\hat{b}_j + \hat{b}_i\hat{a}_j), \tag{B.39}$$

where $\hat{a}$, $\hat{b}$ and $\hat{k}$ are three mutually orthogonal unit vectors.

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